Translational and Rotational Invariance in Networked Dynamical Systems

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***Abstract*—In this paper, we study the translational and rotational ($SE(N)$) invariance properties of locally interacting multi-agent systems. We focus on a class of networked dynamical systems, in which the agents have local pairwise interactions, and the overall effect of the interaction on each agent is the sum of the interactions with other agents. We show that such systems are $SE(N)$-invariant if and only if they have a special, *quasi-linear* form. The $SE(N)$-invariance property, sometimes referred to as left invariance, is central to a large class of kinematic and robotic systems. When satisfied, it ensures independence to global reference frames. In an alternate interpretation, it allows for integration of dynamics and computation of control laws in the agents' own reference frames. Such a property is essential in a large spectrum of applications, e.g., navigation in GPS-denied environments. Because of the simplicity of the quasi-linear form, this result can impact ongoing research on design of local interaction laws. It also gives a quick test to check if a given networked system is $SE(N)$-invariant.**

***Index Terms*—translational and rotational invariance, networked systems, pairwise interaction.**

I. Introduction

In this paper we present necessary and sufficient conditions for a multi-agent system with pairwise interactions to be invariant under translations and rotations of the inertial frame in which the dynamics are expressed (i.e. $SE(N)$-invariant). This kind of invariance allows agents to execute their control laws in their body reference frame [1], [2], [3], using information measured in their body reference frame, without affecting the global evolution of the system. This is critical for any scenario where global information about an agent's reference frame is not readily available, for example coordinating agents underground, underwater, inside of buildings, in space, or in any GPS denied environment [4], [5], [6].

We assume that the agents are kinematic in N-dimensional Euclidean space, and their control laws are computed as sums over all pairwise interactions with their neighbors. We prove that the dynamics are $SE(N)$-invariant if and only if the pairwise interactions are *quasi-linear*, meaning linear in the difference between the states of the two agents, multiplied by a scalar gain which depends only on the distance between the states of the two agents. This result can be used as a test (does a given multi-agent controller require global information, i.e., a common reference frame known by all agents?), or as a design specification (a multi-agent controller is required that uses only local information represented in the agents' private reference frames, hence only quasi-linear pairwise interactions can be considered). It can also be used to test hypothesis about local interaction laws in biological (e.g., locally interacting cells) and physical systems.

We prove the result for agents embedded in Euclidean space of any dimension, and the result holds for arbitrary graph topologies, including directed or undirected, switching, time varying, and connected or unconnected. We show that many existing multi-agent protocols are quasi-linear and thus $SE(N)$-invariant. Examples include the interactions from the classical n-body problem [7] and most of the existing multi-agent consensus and formation control algorithms, e.g., [8], [9], [10], [11], [12], [13], [14]. In particular, explicit consensus algorithms implemented using local information in the agents' body frames [6] satisfy the $SE(N)$-invariance property, as expected. We also show that some multi-agent interaction algorithms, such as [15], are not $SE(N)$-invariant, and therefore cannot be implemented locally in practice. To further illustrate how the main result relates to the literature, we consider a sub-class of $SE(N)$-invariant (and therefore quasi-linear) pairwise interaction systems, and show that they reach a consensus, using the graph Laplacian to represent the system dynamics and the typical LaSalle's invariance analysis to show convergence. Finally, we extend the $SE(N)$-invariance notion to discrete-time systems, dynamical systems of higher order and systems with switching topologies. Moreover, for a sub-class of discrete-time $SE(N)$-invariant pairwise interaction systems, we show that they reach consensus by exploiting the quasi-linear structure given by the main result.

With a few exceptions [16], [17], [18], [6], the problem of invariance to global reference frames was overlooked in the multi-agent control and consensus literature. In [16], the authors discuss invariance for the particular cases of $SE(2)$ and $SE(3)$ actions, and focus on virtual structures. Rotational and translational invariance is also discussed in [17] for a class of algorithms driving a team of agents to a rigid structure. In [19], [20], [21], [22], the notion of shape-coordinates [23] is considered for multi-agent motion planning, where the global rotation and translation of the group of vehicles are quotient out. Invariance to group actions in multi-agent systems was recently studied in [18], where the authors present a general framework to find all symmetries in a given second-order planar system. The authors' main motivation is to determine changes of coordinates transformations that align the system

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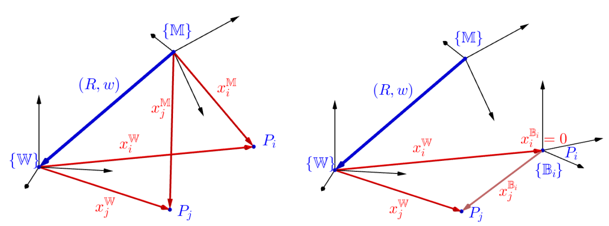
with the symmetry directions and thus aid in stability analysis using LaSalle's principle. This paper is complementary to our work, in the sense that the authors start from a system and find invariants, while in our case we start from an invariance property and find all systems satisfying it. Our results hold for any (finite) dimensional agent state space. Finally, our characterization of invariance is algebraic, and as a result does not require any smoothness assumptions on the functions modeling the interactions. As a result, it can be used for a large class of systems, including discrete-time systems.

Preliminary results from this work were presented in a conference version [24]. The present paper expands on [24] by including all proofs of the main results, as well as new results on the stability of $SE(N)$-invariant systems, switching network topologies, and discrete-time systems. We also provide several new examples with simulations.

II. Significance of $SE(N)$-invariance

In this section we present $SE(N)$-invariance from a geometrical perspective and give two interpretations that prove to be useful for networked agent systems. Formal definitions are provided in Sec. III together with the main result of the paper.

$SE(N)$ is the Special Euclidean group that acts on $\mathbb{R}^N$, i.e., the set of all possible rotations and displacements in $\mathbb{R}^N$. As mentioned before, $SE(N)$-invariance is a property related to reference frames. Consider a global inertial (world) reference frame $\{\mathbb{W}\}$, and another (mobile) reference frame $\{\mathbb{M}\}$, which is related to $\{\mathbb{W}\}$ by the rotation and translation pair $(R, w) \in SE(N)$. Also, consider a networked system with n agents whose states evolve in $\mathbb{R}^N$, and which interact with each other in a pairwise manner, i.e. interaction is point-to-point and may be one-way. Interaction is interpreted as agents measuring the states of their neighbors. Let $x_i^{\mathbb{W}}$ and $x_i^{\mathbb{M}}$ be the state of agent i in reference frames $\{\mathbb{W}\}$ and $\{\mathbb{M}\}$, respectively. (See Fig. 1(a) for an illustration in the case of N=3). The states of agent i in the two references frames are related by $x_i^{\mathbb{W}} = R x_i^{\mathbb{M}} + w$.



(a) (b)

Fig. 1. The diagram in (a) shows the world frame $\{\mathbb{W}\}$, the reference frame $\{\mathbb{M}\}$, two agents i and j and their states in these two frames. The diagram in (b) shows the agents' states expressed in the body frame of agent i.

The relationship between agent i's velocities in the two reference frames is defined by how these are measured and represented. Let ${}^{\mathbb{W}}v_i^{\mathbb{W}}$ and ${}^{\mathbb{W}}v_i^{\mathbb{M}}$ be the velocities measured with respect to the world frame $\{\mathbb{W}\}$ and represented in $\{\mathbb{W}\}$ and $\{\mathbb{M}\}$, respectively. Thus, ${}^{\mathbb{W}}v_i^{\mathbb{W}} = R\, {}^{\mathbb{W}}v_i^{\mathbb{M}}$. On the other hand, agent i's dynamics is ${}^{\mathbb{W}}v_i^{\mathbb{W}} = f_{ij}(x_i^{\mathbb{W}}, x_j^{\mathbb{W}})$, assuming for simplicity that agent i interacts only with agent j.

The notion of $SE(N)$-invariance says that the dynamics of agent i must be the same in all reference frames, i.e. ${}^{\mathbb{W}}v_i^{\mathbb{M}} = f_{ij}(x_i^{\mathbb{M}}, x_j^{\mathbb{M}})$ must hold for all $\{\mathbb{M}\}$. A quick substitution yields $R\, {}^{\mathbb{W}}v_i^{\mathbb{M}} = f_{ij}(R x_i^{\mathbb{M}} + w, R x_j^{\mathbb{M}} + w)$. On the other hand we have $R\, {}^{\mathbb{W}}v_i^{\mathbb{M}} = R f_{ij}(x_i^{\mathbb{M}}, x_j^{\mathbb{M}})$, which implies that $SE(N)$-invariance reduces to $R f_{ij}(x_i^{\mathbb{M}}, x_j^{\mathbb{M}}) = f_{ij}(R x_i^{\mathbb{M}} + w, R x_j^{\mathbb{M}} + w)$ for all values of the states $x_i^{\mathbb{M}}, x_j^{\mathbb{M}}$ and all transformations $(R, w) \in SE(N)$. This is the notion of left invariance that we will define formally in Sec. III. Notice that $SE(N)$-invariance is a basic assumption very common in physical models (i.e. the laws of physics must be the same in all inertial reference frames). In the context of differential geometry, this intuition is formalized by the notion of left-invariance of vector fields.

In the context of networked systems, each agent maintains an individual mobile reference frame. If the reference frames of all agents coincide, then they achieve global localization (this may be implemented using GPS, SLAM, etc.). However, if we desire a truly distributed system, then the agents must maintain local reference frames, which are not synchronized with each other, and be able to compute their own individual control laws in their own local frames. A special choice of mobile reference frames are the body frames associated with each agent i $\{\mathbb{B}_i\}$, (see Fig. 1(b)). The agents measure (using on-board sensors such as cameras) and express the states of all their neighbors in their own individual reference frames $\{\mathbb{B}_i\}$. Consequently, if the system is $SE(N)$-invariant, then the agents can compute their individual control laws (their velocities) in their own body frames, without the need of a predefined global reference. Therefore, we consider that, in practice, $SE(N)$-invariance is a very important property of distributed networked systems.

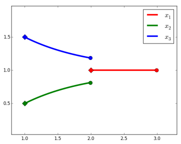
Another interpretation of $SE(N)$-invariance is related to the networked system's behavior, i.e. the agents' trajectories. The invariance property implies that the system produces the same trajectories in any two reference frames. The trajectories of an agent have the same shape and scale (they are isometric) and are related by the transformation between the two reference frames. Fig. 2 shows an example of two sets of trajectories generated by an $SE(2)$-invariant and a non-$SE(2)$-invariant systems in two reference frames, respectively.

III. Definitions and Main Result

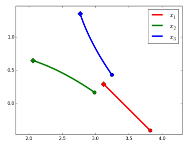
In this section, we introduce the notions and definitions used throughout the paper. The main result of the paper is stated at the end of the section.

For a set S, we use $|S|$ to denote its cardinality. The sets $\mathbb{R}_{\geq a}$ and $\mathbb{Z}_{\geq p}$ represent the interval $[a, \infty)$ and $\{p, p+1, \ldots\}$, where $a \in \mathbb{R}$ and $p \in \mathbb{Z}$, respectively. The notation $\triangleq$ denotes a definition. The canonical basis for the Euclidean space of dimension N, denoted by $\mathbb{R}^N$, is $\{e_1, \ldots, e_N\}$. We use I_N and $\mathbf{1}_N$ to denote the $N \times N$ identity matrix and the $N \times 1$ vector of ones, respectively. The special orthogonal group acting on $\mathbb{R}^N$ is denoted by $SO(N)$. Similarly, $SE(N)$ represents the special Euclidean group of rotations and translations acting on $\mathbb{R}^N$. Throughout the paper, $\|\cdot\|$ refers to the Euclidean norm. The Kronecker product of two matrices is denoted by $\otimes$.

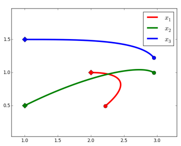
Given a directed graph G, we use $V(G)$ and $E(G) \subseteq V(G) \times V(G)$ to denote its sets of nodes and edges, respectively. An edge $(i, j) \in E(G)$ is interpreted as starting from



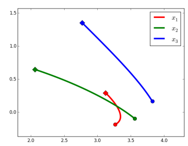
(a) Trajectories of a $SE(2)$-invariant system as seen from $\{\mathbb{W}_1\}$



(b) Trajectories of a $SE(2)$-invariant system as seen from $\{\mathbb{W}_2\}$



(c) Trajectories of a non-$SE(2)$-invariant system as seen from $\{\mathbb{W}_1\}$



(d) Trajectories of a non-$SE(2)$-invariant system as seen from $\{\mathbb{W}_2\}$

Fig. 2. The figure shows the trajectories of two systems in two reference frames $\{\mathbb{W}_1\}$ and $\{\mathbb{W}_2\}$, which are related by a rotation $R(\pi/4)$ in clockwise direction and a translation $w = [1, 1]^T$. Clearly, the trajectories generated by the $SE(2)$-invariant system have the same shape and are related by (R, w), (a) and (b). The shape of the trajectories generated by the non-$SE(2)$-invariant system are different in the two reference frames, (c) and (d).

i and ending at j. An edge starting at i is called an outgoing edge of i, while an edge ending at i is called an incoming edge of i. Given a node $i \in V(G)$, $\mathcal{N}_i^{\rightarrow}$ stands for the set of outgoing neighbors of i, i.e. $\mathcal{N}_i^{\rightarrow} = \{j \in V(G) | (i,j) \in E(G)\}$. Similarly, $\mathcal{N}_i^{\leftarrow} = \{j \in V(G) | (j,i) \in E(G)\}$ represents the set of incoming neighbors of i.

Definition III.1 ($SE(N)$-invariant function)**.** *A function $f : \mathbb{R}^N \times \cdots \times \mathbb{R}^N \to \mathbb{R}^N$ is said to be $SE(N)$-invariant if for all $R \in SO(N)$ and all $x_1, , x_p, w \in \mathbb{R}^N$*

$$Rf(x_1, \ldots, x_p) = f(Rx_1 + w, \ldots, Rx_p + w). \tag{1}$$

Definition III.2 (Pairwise Interaction System)**.** *A continuous-time pairwise interaction system* [1] *is a pair (G, F), where G is a graph and $F = \{f_{ij} \mid f_{ij} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N, (i,j) \in E(G)\}$ is a set of functions associated to its edges. Each $i \in V(G)$ labels an agent, and a directed edge (i, j) indicates that node i interacts with (measures the state of) node j. The dynamics of each agent are described by*

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij}(x_i, x_j), \tag{2}$$

where f_{ij} defines the influence (interaction) of j on i.

We denote the *total interaction* on agent $i \in V(G)$ by

$$S_i(x_1, \ldots, x_{|V(G)|}) = \sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij}(x_i, x_j). \tag{3}$$

Definition III.3 ($SE(N)$-Invariance)**.** *A pairwise interaction system (G, F) is said to be $SE(N)$-invariant if, for all $i \in V(G)$, the total interaction functions S_i are $SE(N)$-invariant.*

Definition III.4 (Quasi-linear function)**.** *A function $f : \mathbb{R}^N \to \mathbb{R}^N$ is said to be quasi-linear if there is a function $k : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $f(x) = k(\|x\|)x$, for all $x \in \mathbb{R}^{\mathbb{N}}$.*

[1]For continuous-time systems, we assume that f_{ij} and k_{ij} are Lipschitz continuous for all $(i, j) \in E(G)$.

Definition III.5 (Quasi-linear interaction system)**.** *A pairwise interaction system (G, F) is said to be quasi-linear if the total interaction S_i of each agent i is a sum of quasi-linear functions. Formally, for all $i \in V(G)$*

$$S_i = \sum_{j \in \mathcal{N}_i^{\rightarrow}} k_{ij}(\|x_j - x_i\|)(x_j - x_i), \tag{4}$$

where $k_{ij} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ are scalar gain functions[1] *for all $j \in \mathcal{N}_i^{\rightarrow}$ and $N \geq 3$.*

Remark III.6. *The definition of quasi-linearity for pairwise interaction systems is a statement about the overall dynamics of agents. Specifically, Def. III.5 does not imply that the local pairwise interaction functions f_{ij} are themselves quasi-linear functions. See Ex. VIII.1.*

The main result of this paper can be stated as follows:

Theorem III.7. *A pairwise interaction system (G, F) is $SE(N)$-invariant if and only if it is quasi-linear.*

Remark III.8. *The pairwise interaction form of the systems considered in this paper is a fundamental assumption needed to obtain the main result, Thm. III.7. To illustrate this, consider a system with three agents and the total interaction function $S_1(x_1, x_2, x_3) = \|x_2 - x_1\| (x_3 - x_2)$ of agent 1, which captures a three-way interaction among the agents. By Def. III.1 S_1 is $SE(N)$-invariant. Indeed, for all $(R, w) \in SE(N)$*

$$\begin{aligned} &RS_1(x_1, x_2, x_3) = \|x_2 - x_1\| R(x_3 - x_2) \\ &= \|Rx_2 + w - (Rx_1 + w)\| (Rx_3 + w - (Rx_1 + w)) \\ &= S_1(Rx_1 + w, Rx_2 + w, Rx_3 + w). \end{aligned}$$

However, S_1 cannot be written as a sum of quasi-linear functions.

Remark III.9. *Since $SE(N)$-invariance is a property of reference frames, it does not imply anything about the stability of the system. The converse does not hold either. Therefore, we can have unstable $SE(N)$-invariant systems and stable systems which are not $SE(N)$-invariant. See Sec. VIII.*

Remark III.10. *Note that we do not impose any restrictions on the graph G, i.e., the results hold even if G is disconnected. Also, the functions in F may not be related to each other, i.e., we do not assume any functional constraints between local interactions functions. Symmetry properties, such as $f_{ij} = f_{ji}$ and $\sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij} + \sum_{j \in \mathcal{N}_i^{\leftarrow}} f_{ji} = 0$, are not needed.*

The main result of the paper (Thm. III.7) can be regarded as a characterization of $SE(N)$-invariant functions arising from pairwise interaction systems. We establish their structure in Sec. IV, where we show that all local interaction functions are quasi-linear functions with additional affine terms, whose sums over each agent's neighbors vanish. Thus, it follows that the total interaction functions are quasi-linear. As an intermediate step of the proof, we show that functions commuting with $SO(N)$ are quasi-linear. We provide stability results on $SE(N)$-invariant systems in Sec. VI. Finally, in Sec. VII, we include extensions of Thm. III.7 to discrete-time systems, higher order systems and switching topologies.

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IV. Characterizing the Centralizers of $SO(N)$

In this section, we prove that functions which commute with $SO(N)$ are quasi-linear, which generalizes the well-known result for linear functions [25]. We establish the general case using induction on $N \geq 3$. The case $N = 2$ is treated separately in App. X.

Let $T = \{f : \mathbb{R}^N \to \mathbb{R}^N\}$ be the set of all transformations (not necessarily bijective) acting on $\mathbb{R}^N$. T is the *transformation monoid* with respect to function composition.

Definition IV.1 (Centralizer)**.** *Let A be a sub-semigroup of T. The* centralizer *of A with respect to T is denoted by $C_T(A)$ and is the set of all elements of T that commute with all elements of A, i.e. $C_T(A) = \{f \in T | fg = gf, \forall g \in A\}$.*

The centralizer $C_T(A)$ is a submonoid of T and can be interpreted as the set of transformations invariant with respect to all transformations in A. In other words, the action of $f \in C_T(A)$ on $\mathbb{R}^N$ and then transformed by $g \in A$ is the same as the action of f on the transformed space $g(\mathbb{R}^N)$.

Note that the set of all quasi-linear functions is a submonoid of T, which will be denoted by $QL(N)$. We implicitly identify the elements of $SO(N)$ with linear maps acting on $\mathbb{R}^N$, and commutativity is defined with respect to function composition.

Before we proceed, we provide two lemmas that are used in subsequent proofs. The following lemma, whose proof is straightforward and omitted, shows the intuitive fact that the only vector invariant under all rotations is the null vector.

Lemma IV.2. *Let $x \in \mathbb{R}^N$. If $Rx = x$ for all $R \in SO(N)$, $N \geq 2$, then $x = 0$.*

Lemma IV.3. *If $f = (f_1, \ldots, f_N) : \mathbb{R}^N \to \mathbb{R}^N$ commutes with all elements of $SO(N)$, then $x^T f(x) = \|x\| f_1(\|x\| e_1)$, for all $x \in \mathbb{R}^N$.*

Proof. Let $x \in \mathbb{R}^N$ and $R \in SO(N)$ such that $x = R^T \|x\| e_1$. It follows that $f(x) = f(R^T \|x\| e_1) = R^T f(\|x\| e_1)$. Finally, $x^T f(x) = x^T R^T f(\|x\| e_1) = (Rx)^T f(\|x\| e_1) = \|x\| e_1^T f(\|x\| e_1) = \|x\| f_1(\|x\| e_1)$. □

The following three lemmas establish the base case $N = 3$ of the induction argument used in the proof of Thm. IV.7.

Lemma IV.4. *If $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ with $\|u\| = 1$ and $u \neq \pm e_1$, then*
$$R_u = \begin{bmatrix} u_1 & 0 & -\sqrt{u_2^2 + u_3^2} \\ u_2 & \frac{u_3}{\sqrt{u_2^2+u_3^2}} & \frac{u_1 u_2}{\sqrt{u_2^2+u_3^2}} \\ u_3 & -\frac{u_2}{\sqrt{u_2^2+u_3^2}} & \frac{u_1 u_3}{\sqrt{u_2^2+u_3^2}} \end{bmatrix} \in SO(3).$$

Proof. The matrix satisfies $R_u R_u^T = I_3$ and $\det(R_u) = 1$, and thus it is a rotation matrix in $SO(3)$. □

Lemma IV.5. *Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. If $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ commutes with all elements of $SO(3)$, then*
$$\begin{aligned} f_1(x) &= -f_1(-x_1, -x_2, x_3) & (5)\\ f_1(x) &= -f_1(-x_1, x_2, -x_3) & (6)\\ f_2(x) &= f_1(x_2, -x_1, x_3) & (7)\\ f_3(x) &= f_1(x_3, x_2, -x_1) & (8) \end{aligned}$$

Proof. The above relationships can be obtained using 90° rotation matrices around the axes e_1, e_2 and e_3. □

Proposition IV.6. *The centralizer of $SO(3)$ with respect to T is the monoid of quasi-linear functions $QL(3)$.*

Proof. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x \neq \alpha e_1$, $\alpha \in \mathbb{R}$ and $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$. Let $u = \frac{x}{\|x\|}$ and R_u as in Lemma IV.4, we have $x = R_u \|x\| e_1$ and $u_i = \frac{x_i}{\|x\|}$. Using the commutation property we obtain $f(x) = f(R_u \|x\| e_1) = R_u f(\|x\| e_1)$ and writing the equation for f_1, it follows that
$$f_1(x) = u_1 f_1(\|x\| e_1) - \sqrt{u_2^2 + u_3^2}\, f_3(\|x\| e_1). \qquad (9)$$

Using (8) from Lemma IV.5, we have $f_3(\|x\|, 0, 0) = f_1(0, 0, -\|x\|)$. On the other hand, it follows from (5) that $f_1(0, 0, \alpha) = -f_1(0, 0, \alpha)$, which implies $f_1(0, 0, \alpha) = 0$ for all $\alpha \in \mathbb{R}$. It follows that $f_3(\|x\| e_1) = 0$ for all $x \in \mathbb{R}^3$, $x \neq \alpha e_1$ and $\alpha \in \mathbb{R}$. Thus, (9) can be simplified to
$$f_1(x) = x_1 f_1(\|x\| e_1) \|x\|^{-1} = x_1 k(\|x\|), \qquad (10)$$
where $k : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is $k(\alpha) \triangleq \frac{f_1(\alpha e_1)}{\alpha}$, $\alpha \geq 0$. The general form of $f(x) = k(\|x\|)x$ is obtained using (7) and (8).

The case $x = 0$ follows easily from Lemma IV.2, because it implies $f(0) = 0$. The remaining case $x = \alpha e_1$ is trivial; $f(\alpha e_1) = [f_1(\alpha e_1)\ f_2(\alpha e_1)\ f_3(\alpha e_1)]^T = [\alpha \frac{f_1(\alpha e_1)}{\alpha}\ 0\ 0]^T = k(\|x\|)x$, where $f_2(\alpha e_1) = 0$ and $f_3(\alpha e_1) = 0$ follow from (7), (6), and (8), (5), respectively.

Conversely, if $f \in QL(N)$, then $Rf(x) = R(k(\|x\|)x) = k(\|Rx\|)Rx = f(Rx)$, where $R \in SO(3)$. Thus, we have $f \in C_T(SO(3))$, which concludes the proof. □

Theorem IV.7. *The centralizer of $SO(N)$ with respect to T is the monoid of quasi-linear functions $QL(N)$, for all $N \geq 3$.*

Proof. The proof follows from an induction argument with respect to N. The base case is established by Prop. IV.6. To simplify the notation, given a vector $x = (x_1, \ldots, x_N)$ we will denote by $x_{i:j}$, $i < j$, the sliced vector $(x_i, \ldots, x_j) \in \mathbb{R}^{j-i+1}$.

The induction step: Let $x \in \mathbb{R}^{N+1}$, $x \neq 0$, and $R_1 = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$, $R_2 = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$, where $R \in SO(N)$. Using R_1, it follows that $Rf_{1:N}(x_{1:N}, x_{N+1}) = f_{1:N}(Rx_{1:N}, x_{N+1})$. Applying the induction hypothesis, we obtain
$$f_{1:N}(x_{1:N}, x_{N+1}) = k_1(\|x_{1:N}\|, x_{N+1}) x_{1:N}\,. \qquad (11)$$
Similarly, using R_2 we have $Rf_{2:N+1}(x_1, x_{2:N+1}) = f_{2:N+1}(x_1, Rx_{2:N+1})$ and obtain
$$f_{2:N+1}(x_1, x_{2:N+1}) = k_2(\|x_{2:N+1}\|, x_1) x_{2:N+1}\,. \qquad (12)$$
Equating (11) and (12) for f_2 and assuming w.l.o.g. $x_2 \neq 0$, we get a constraint between the two gains
$$k_2(\|x_{2:N+1}\|, x_1) = k_1(\|x_{1:N}\|, x_{N+1})\,. \qquad (13)$$
Thus, we obtain f_{N+1} in terms of the gain k_1 by using the last equality from (12) and (13) to substitute k_2 for k_1
$$f_{N+1}(x_1, \ldots, x_{N+1}) = k_1(\|x_{1:N}\|, x_{N+1}) x_{N+1}\,. \qquad (14)$$
Finally, putting all the components of f obtained from (11)

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and (14) together and left multiplying it by x^T, we get

$$x^T f(x) = \sum_{i=1}^{N+1} x_i^2 k_1(\|x_{1:N}\|, x_{N+1}) \\ = \|x\|^2 k_1(\|x_{1:N}\|, x_{N+1}) = \|x\| f_1(\|x\| e_1),$$

where the last equality follows from Lemma IV.3. It follows that $k_1(\|x_{1:N}\|, x_{N+1}) = \frac{f_1(\|x\|e_1)}{\|x\|} \triangleq k(\|x\|)$. Thus, $f(x) = k(\|x\|)x$ or equivalently $f \in C_T(SO(N))$.

Conversely, we have $QL(N) \subseteq C_T(SO(N))$ (see proof of Prop. IV.6). □

V. $SE(N)$-INVARIANT FUNCTIONS

In this section, we use the result from the previous section $C_T(SO(N)) = QL(N)$ in order to characterize $SE(N)$-invariant functions that arise from pairwise interaction systems.

Proposition V.1. *A function $h(x_1, x_2) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is $SE(N)$-invariant if and only if h is quasi-linear in $x_2 - x_1$.*

Proof. Trivially, a quasi-linear function $h(x_1, x_2) = k(\|x_2 - x_1\|)(x_2 - x_1)$ is $SE(N)$-invariant. Conversely, if $R = I_N$ and $w = -x_2$, then $h(x_1, x_2) = h(x_1 - x_2, x_2 - x_2) = h(x_1 - x_2, 0) \triangleq \hat{h}(x_2 - x_1)$. Let $x \in \mathbb{R}^N$ and $R \in SO(N)$, it follows that $R\hat{h}(x) = Rh(-x, 0) = h(-Rx, 0) = \hat{h}(Rx)$. Since $\hat{h}$ commutes with all elements of $SO(N)$ it follows by Thm. IV.7 that it is quasi-linear. Thus, we have $h(x_1, x_2) = \hat{h}(x_2 - x_1) = k(\|x_2 - x_1\|)(x_2 - x_1)$. □

Lemma V.2. *Let $h_1, h_2 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$. Then $S(x_0, x_1, x_2) = h_1(x_0, x_1) + h_2(x_0, x_2)$ is an $SE(N)$-invariant function if and only if there exists $k_1(\cdot)$ and $k_2(\cdot)$ such that for all $x_0, x_1, x_2 \in \mathbb{R}^N$ we have*

$$h_1(x_0, x_1) = h_1(x_0, x_0) + k_1(\|x_1 - x_0\|)(x_1 - x_0) \quad (15)$$

$$h_2(x_0, x_2) = h_2(x_0, x_0) + k_2(\|x_2 - x_0\|)(x_2 - x_0) \quad (16)$$

and $h_1(x_0, x_0) + h_2(x_0, x_0) = 0$.

Proof. It is easy to show that if S is the sum of functions satisfying (15), (16) and the zero-sum constraint, then S is $SE(N)$-invariant. Conversely, let $f_1(a, b) = h_1(a, b) + h_2(a, a)$ and $f_2(a, b) = h_1(a, a) + h_2(a, b)$, where $f_1, f_2 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and $a, b \in \mathbb{R}^N$. It follows immediately that f_1 and f_2 are $SE(N)$-invariant, because $h_1(x_0, x_1) + h_2(x_0, x_2)$ is $SE(N)$-invariant. Therefore, we have by Prop. V.1 that $f_1(a, b) = k_1(\|b - a\|)(b - a)$ and $f_2(a, b) = k_2(\|b - a\|)(b - a)$. Choosing $a = b$ in any of the previous two equations, we obtain $h_1(a, a) + h_2(a, a) = 0$. Finally, $h_1(a, b) = -h_2(a, a) + f_1(a, b) = h_1(a, a) + k_1(\|b - a\|)(b - a)$ and $h_2(a, b) = -h_1(a, a) + f_2(a, b) = h_2(a, a) + k_2(\|b - a\|)(b - a)$. □

Lemma V.3. *Let $h_1, \ldots, h_p : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $p \in \mathbb{Z}_{\geq 2}$. Then $S(x_0, \ldots, x_p) = \sum_{i=1}^p h_i(x_0, x_i)$ is an $SE(N)$-invariant function if and only if for all $i \in \{1, \ldots, p\}$ there exists $k_i(\cdot)$ such that for all $x_0, x_1, \ldots, x_p \in \mathbb{R}^N$ we have*

$$h_i(x_0, x_i) = h_i(x_0, x_0) + k_i(\|x_i - x_0\|)(x_i - x_0) \quad (17)$$

$$\sum_{i=1}^p h_i(x_0, x_0) = 0\,. \quad (18)$$

Proof. As before, the quasi-linearity of S, which follows from (17) and (18), trivially implies its $SE(N)$-invariance. We will prove the converse by induction with respect to p. The base step $p = 2$ is established by Lemma V.2. For the induction step, we assume that Lemma V.3 holds for p and we must show that it also holds for $p + 1$.

Let $x_{p+1} = x_1$ and define $h'_1(x_0, x_1) = h_1(x_0, x_1) + h_{p+1}(x_0, x_1)$. Clearly $h'_1(x_0, x_1) + \sum_{i=2}^p h_i(x_0, x_i)$ is an $SE(N)$-invariant function and by the induction hypothesis we have for all $i \in \{2, \ldots, p\}$

$$h_i(x_0, x_i) = h_i(x_0, x_0) + k_i(\|x_i - x_0\|)(x_i - x_0)$$
$$h'_1(x_0, x_1) = h'_1(x_0, x_0) + k'_1(\|x_1 - x_0\|)(x_1 - x_0) \\ = h_1(x_0, x_0) + h_{p+1}(x_0, x_0) + k'_1(\|x_1 - x_0\|)(x_1 - x_0)$$

and $h'_1(x_0, x_0) + \sum_{i=2}^p h_i(x_0, x_0) = \sum_{i=1}^{p+1} h_i(x_0, x_0) = 0$.

Similarly, let $x_{p+1} = x_2$ and define $h'_2(x_0, x_2) = h_2(x_0, x_2) + h_{p+1}(x_0, x_2)$. Using the same argument as before, we obtain $h_1(x_0, x_1) = h_1(x_0, x_0) + k_1(\|x_1 - x_0\|)(x_1 - x_0)$. Substituting h_1 in the expression of h'_1 and solving for h_{p+1} we have $h_{p+1}(x_0, x_{p+1}) = h'_1(x_0, x_{p+1}) - h_1(x_0, x_{p+1}) = h_{p+1}(x_0, x_0) + k_{p+1}(\|x_{p+1} - x_0\|)(x_{p+1} - x_0)$, where $k_{p+1} = k'_1 - k_1$. This concludes the proof. □

We conclude this section with a characterization theorem of the total interaction functions of pairwise interaction systems.

Theorem V.4. *Let $S(x_0, x_1, \ldots, x_p) = \sum_{j=1}^p h_j(x_0, x_j)$, where $h_j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and $p \geq 1$. Then S is $SE(N)$-invariant if and only if it is the sum of quasi-linear functions in $x_j - x_0$, $j \in \{1, \ldots, p\}$, that is $S = \sum_{j=1}^p k_j(\|x_j - x_0\|)(x_j - x_0)$, where $k_j : \mathbb{R}_{\geq 0} \to \mathbb{R}$.*

Proof. Let $S(x_0, \ldots, x_p) = \sum_{j=1}^p h_j(x_0, x_j)$ be an $SE(N)$-invariant function, it follows from Lemma V.3 that there exists $k_j(\cdot)$ for all $j \in \{1, \ldots, p\}$, such that

$$S = \sum_{j=1}^p h_j(x_0, x_0) + \sum_{j=1}^p k_j(\|x_j - x_0\|)(x_j - x_0) \\ = \sum_{j=1}^p k_j(\|x_j - x_0\|)(x_j - x_0),$$

where the last equality follows from (18) of Lemma V.3, which says that the sum of all affine terms must vanish.

Conversely, let $S = \sum_{j=1}^p k_j(\|x_j - x_0\|)(x_j - x_0)$, then S is $SE(N)$-invariant, i.e. for all $(R, w) \in SE(N)$

$$RS = \sum_{j=1}^p k_j(\|x_j - x_0\|)\ R\,(x_j - x_0) \\ = \sum_{j=1}^p k_j(\|Rx_j + w - (Rx_0 + w)\|)(Rx_j + w - (Rx_0 + w)) \\ = S(Rx_0 + w, Rx_1 + w, \ldots, Rx_p + w),$$

where we used the fact that $\|Rx\| = \|x\|$ for all $R \in SO(N)$ and $x \in \mathbb{R}^N$. The proof is now complete. □

Thm. III.7 follows immediately from Thm. V.4, since we can apply Thm. V.4 on the total interaction function S_i of any agents i, where p, x_0 and $h_j(x_0, x_j)$, $j \in \{1, \ldots, p\}$, correspond to $|\mathcal{N}_i^{\rightarrow}|$, x_i and $f_{ij}(x_i, x_j)$, $j \in \mathcal{N}_i^{\rightarrow}$, respectively.

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Remark V.5. *Theorem III.7 is stated in terms of total interaction functions, independent of a notion of dynamics, which has two benefits: (1) it greatly expands the applicability of the result to other cases (Sec. VII), and (2) we do not need to assume any smoothness conditions on the functions, such as continuity or differentiability.*

VI. Stability of $SE(N)$-invariant systems

In this section, we explore the stability of $SE(N)$-invariant pairwise interaction systems, showing that a subclass of such systems converges to a consensus state (one in which all agents' states are equal). The stability result exploits the structure of $SE(N)$-invariant systems imposed by Thm. III.7 and some additional constraints on the connectivity of the interaction graph and local interaction functions.

Before we state the stability theorem, we prove a lemma connecting the Laplacian matrix of a graph with the convergence rate of the systems towards the equilibria set.

Lemma VI.1. *Let $\mathcal{L}$ be a $n \times n$ real symmetric positive semidefinite matrix with eigenvalues $\lambda_n \geq \ldots \geq \lambda_2 > \lambda_1 = 0$ and $\mathbf{1}_n$ be the right eigenvector corresponding to the eigenvalue $\lambda_1 = 0$. For all $x \in \mathbb{R}^{Nn}$, $N > 2$, such that $(\mathbf{1}_n^T \otimes I_N)x = 0$, we have $x^T(\mathcal{L} \otimes I_N)x \geq \lambda_2(\mathcal{L}) \|x\|^2$.*

Proof. The spectrum of the Kronecker product of two matrices is composed of the pairwise products of their eigenvalues. Thus, $\mathcal{L} \otimes I_N$ has the same eigenvalues as $\mathcal{L}$. The inequality follows from the Courant-Fisher theorem [8], [26]. □

Theorem VI.2. *Let (G, F) be a continuous-time pairwise-interaction system that satisfies the following properties:*

1) *(G, F) is $SE(N)$-invariant;*
2) *G is strongly connected;*
3) *(G, F) is balanced, i.e. for all agents i and $x_i, x_j \in \mathbb{R}^N$*

$$\sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij}(x_i, x_j) + \sum_{j \in \mathcal{N}_i^{\leftarrow}} f_{ji}(x_j, x_i) = 0 \tag{19}$$

4) *positivity – for all $(i, j) \in E(G)$ and $x_i \neq x_j \in \mathbb{R}^N$*

$$(x_j - x_i)^T \left(f_{ij}(x_i, x_j) - f_{ij}(x_i, x_i)\right) > 0. \tag{20}$$

The consensus set $\Omega(\bar{x}(0)) = \{x | x_i = \bar{x}(0), \forall i \in V(G)\}$ is globally asymptotically stable, where $x = [x_1^T, \ldots, x_n^T]^T$ is the stacked state vector and $\bar{x}(0) = \frac{1}{n}\sum_{i=1}^n x_i(0)$, $n = |V(G)|$. Moreover, $\sigma_{ij} = \lim_{x_i \to x_j} \frac{(x_j - x_i)^T(f_{ij}(x_i,x_j) - f_{ij}(x_i,x_i))}{\|x_j - x_i\|^2}$ exists for each $(i, j) \in E(G)$, and if $\sigma_{ij} > 0$ for all $(i, j) \in E(G)$, then $\Omega(\bar{x}(0))$ is globally exponentially stable.

Proof. The proof uses a Lyapunov function based argument similar to [8, Thm. 3]. We use Thm. III.7 to rewrite the dynamics of the system in quasi-linear form. We proceed to define a weighted Laplacian matrix, where the weights are dependent on the agents' states, which is the main difference from the proof in [8]. Finally, we define a quadratic Lyapunov function and show that the total derivative is upper bounded by the Fiedler value of the Laplacian matrix and thus guarantees global asymptotic stability. The details are presented below.

First, we show that the average state $\bar{x}(t) = \frac{1}{n}\sum_{i \in V(G)} x_i(t)$ is time-invariant. The derivative of $\bar{x}(t)$ is

$$\begin{aligned}\dot{\bar{x}} &= \frac{1}{n}\sum_{i \in V(G)} \sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij}(x_i, x_j) = \frac{1}{n}\sum_{(i,j) \in E(G)} f_{ij}(x_i, x_j) \\ &= \frac{1}{2n}\sum_{i \in V(G)} \Big(\sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij}(x_i, x_j) + \sum_{j \in \mathcal{N}_i^{\leftarrow}} f_{ji}(x_j, x_i)\Big) = 0,\end{aligned}$$

where the third equality follows from writing the sum of all local interaction functions in two ways, using the incoming and outgoing edges. The last equality follows from the assumption that (G, F) is balanced.

Let $\delta(t) = x(t) - \mathbf{1}_n \otimes \bar{x}(0)$ be the *disagreement* vector. The next step is to show that the *disagreement space* spanned by δ is orthogonal to the consensus space

$$\begin{aligned}(\mathbf{1}_n^T \otimes D_\alpha)\delta(t) &= (\mathbf{1}_n^T \otimes D_\alpha)x(t) - (\mathbf{1}_n^T \otimes D_\alpha)(\mathbf{1}_n \otimes \bar{x}(0)) \\ &= D_\alpha\left(n\bar{x}(t) - n\bar{x}(0)\right) = 0,\end{aligned}$$

where $\alpha \in \mathbb{R}^n$ and $D_\alpha = \mathrm{diag}(\alpha)$. The last equality above holds due to the conservation of the average state.

Next, we use $SE(N)$-invariance to rewrite the system's dynamics in the quasi-linear form given by Thm. V.4. Let $L(x)$ denote the $n \times n$ weighted Laplacian matrix of (G, F),

$$L_{ij} = \begin{cases} \sum_{p \in \mathcal{N}_i^{\rightarrow}} k_{ip}(\|x_i - x_p\|) & \text{for } i = j \\ -k_{ij}(\|x_i - x_j\|) & \text{for } i \neq j \text{ and } (i,j) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

where $\sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij}(x_i, x_j) = \sum_{j \in \mathcal{N}_i^{\rightarrow}} k_{ij}(\|x_j - x_i\|)(x_j - x_i)$, for all $i, j \in V(G)$. The positivity assumption in (20) implies that $k_{ij}(a) > 0$ for all $(i, j) \in E(G)$ and $a > 0$.

Using the Laplacian, the system dynamics may be written in the compact form $\dot{x} = -(L(x) \otimes I_N)x$. Also, because $k_{ij}(\|x_i - x_j\|) = k_{ij}(\|x_i + \alpha - (x_j + \alpha)\|)$, we have that $L(x) = L(x + (\mathbf{1}_n \otimes \alpha))$, for all $\alpha \in \mathbb{R}^N$, and $L(x) = L(\delta)$. Moreover, the dynamics of the *disagreement* vector is

$$\dot{\delta} = \dot{x} = -(L(x) \otimes I_N)x \tag{21}$$

$$= -(L(\delta) \otimes I_N)\delta + (L(\delta) \otimes I_N)(\mathbf{1}_n \otimes \bar{x}(0)) \tag{22}$$

$$= -(L(\delta) \otimes I_N)\delta, \tag{23}$$

where the second term in (22) vanishes, because $\mathbf{1}_n$ is a right eigenvector of $L(\delta)$.

Let $\hat{L}(x) = \frac{1}{2}(L(x) + L^T(x))$ be the Laplacian matrix of the mirror graph of G, i.e. the graph with both the edges of G and the reversed edges of G. Notice that $\hat{L}(x)$ is symmetric and $x^T\hat{L}(x)x = \frac{1}{2}(x^T L(x)x + x^T L^T(x)x) = x^T L(x)x$.

Consider the Lyapunov function $V(\delta) = \frac{1}{2}\|\delta\|^2$, which is trivially positive definite and radially unbounded. The total derivative of $V(\cdot)$ is

$$\dot{V}(\delta) = \delta^T\dot{\delta} = -\delta^T(L(\delta) \otimes I_N)\delta = -\delta^T(\hat{L}(\delta) \otimes I_N)\delta \tag{24}$$

$$\leq -\lambda_2(\hat{L}(\delta))\|\delta\|^2, \tag{25}$$

where $\lambda_2(\hat{L}(\delta))$ is the Fiedler value (second smallest eigenvalue) of $\hat{L}(\delta)$. The inequality in (25) follows from Lemma VI.1, because G is balanced and thus $\hat{L}(x)\mathbf{1}_n = \frac{1}{2}(L(x)\mathbf{1}_n + L(x)^T\mathbf{1}_n) = 0$.

The total derivative of the Lyapunov function $\dot{V}(\delta)$ is zero if and only if either: (1) δ is zero, or (2) G is not strongly connected. However, the positivity condition (20) implies that G is strongly connected for all $\delta \neq 0$. Since, $\delta = 0$ implies $\dot{\delta} =$

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0 it follows from LaSalle's invariance principle that $\delta^* = 0$ is globally asymptotically stable. It follows that $x^* = \mathbf{1}_n \otimes \bar{x}(0)$ and $\Omega(\bar{x}(0))$ is globally asymptotically stable.

Lastly, we have that $\sigma_{ij} = \lim_{a\to 0} k_{ij}(a) = k_{ij}(0)$ exists for all $(i, j) \in E(G)$ due to the continuity of k_{ij}, where $a = \|x_j - x_i\|$. Since, $\delta(t)$ is differentially continuous in t and a convergent trajectory, and k_{ij} are also continuous, it follows that $\lambda_2(L(\delta(t)))$ is also continuous in t [27] on $\mathbb{R} \cup \{\infty\}$ and its image is a compact set Λ_2. The compactness of Λ_2 implies that it admits a minimum value. If $\sigma_{ij} > 0$, then all values in Λ_2 are positive due to (20). In particular, it follows that $\min \Lambda_2 > 0$. We can then upper bound the quantity in (25) by $\dot{V} \le -\min \Lambda_2 \|\delta\|^2$, that implies $\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\| \le -\min \Lambda_2 \|\delta\|$. Therefore, $\Omega(\bar{x}(0))$ is globally exponentially stable. □

VII. Extensions

The main result in Sec. III is stated for first order (kinematic) continuous-time dynamics. Here we discuss extensions to discrete-time and higher order dynamics, and system with switching and time-varying graph topologies.

A. Discrete-time systems

A discrete-time pairwise interaction system can be defined by replacing differentiation ($\dot{x}_i$) with one-step difference ($\Delta x_i(t) = x_i(t+1) - x_i(t)$) in (2) of Def. III.2. The definitions of the total interaction function and $SE(N)$-invariance remain unchanged, (see (3) of Def. III.2 and Def. III.3, respectively).

The main result, Thm. III.7, holds for discrete-time systems as well. However, the stability results need to be adjusted.

Lemma VII.1. *Let (X, d) be a complete metric space and $(T_n)_{n\ge 0}$ be a sequence of Lipschitz continuous functions such that all admit a Lipschitz constants $q < 1$. Define the sequence $x_{n+1} = T_n(x_n)$. If all maps T_n have the same fixed point $x^* \in X$, then for all $x_0 \in X$ we have $x_n \to x^*$.*

Proof. First note that all maps T_n are contractions, because $q < 1$. Thus, by the contraction mapping theorem, all T_n have a unique fixed point x^*. Moreover, the Lipschitz inequality $d(T_n(x), x^*) = d(T_n(x), T_n(x^*)) \le q d(x, x^*)$ holds for all $n \ge 0$ and $x \in X$. It follows by induction that $d(x_n, x^*) \le q^n d(x_0, x^*)$, for all $n \ge 1$. The base case $n = 1$ follows from the contraction inequality. For the induction step, we again use Lipschitz property, $d(x_{n+1}, x^*) = d(T_n(x_n), x^*) \le q d(x_n, x^*) \le q^{n+1} d(x_0, x^*)$, where in the last inequality we used the induction hypothesis.

Lastly, x_n is a Cauchy sequence, because for all $m, n \ge 0$ $d(x_m, x_n) \le d(x_m, x^*) + d(x^*, x_n) \le (q^m + q^n) d(x_0, x^*)$, where we used the triangle inequality in the first inequality. Therefore, x_n has the unique limit x^*, because X is complete and the distance map d is continuous. □

Definition VII.2. *Let (G, F) be a discrete-time pairwise interaction system and G^T be the transpose graph of G, i.e. the graph with all edges reversed. Denote by S^G and S^{G^T} the vectors of stacked total interaction functions for all agents with interaction graphs G and G^T, respectively. System (G, F) is said to be* forward-backward consistent *if*

$$\left(\mathbf{id} + S^{G^T}\right) \circ \left(\mathbf{id} + S^G\right) = \left(\mathbf{id} + S^G\right) \circ \left(\mathbf{id} + S^{G^T}\right), \quad (26)$$

where $\mathbf{id}$ *is the identity function and* $\circ$ *is function composition.*

Remark VII.3. *The identity function in the terms of* (26) *arises, because the equations of the forward (G) and backward (reversed, G^T) evolution of the system are $x(t+1) = x(t) + S^G(x(t))$ and $x(t+1) = x(t) + S^{G^T}(x(t))$, respectively.*

Remarks VII.4. *Def. VII.2 describes a property about the evolution of a system in two time units, where the edges of the interaction graph are reversed in one of the two time units. Property* (26) *captures the idea that the state the system ends up in is independent of when the edges' reversal occurred.*

The property can also be interpreted in the following way. Consider a network with half-duplex interaction (communication) links and a global switch which changes the direction of all links at the same time. The forward-backward consistency property implies that the state of the network at time t depends only on the initial state and the number of network switches until time t and not the sequence of switches itself.

Yet another way to interpret the property is as a relaxation of time-reversibility. If the two terms in (26) *were equal to the identity function, then the pairwise interaction system (G, F) would be time-reversible and moreover the system could be brought back to the initial state using (G^T, F) with the interaction graph reversed. Therefore, Def. VII.2 can be though of as a relaxation of time-reversibility.*

Theorem VII.5. *Let (G, F) be a discrete-time pairwise interaction system that satisfies the following properties:*

1) *(G, F) is $SE(N)$-invariant;*
2) *G is strongly connected;*
3) *(G, F) is forward-backward consistent, see Def. VII.2;*
4) *strong positivity – for all $(i, j) \in E(G)$*

$$\inf_{x_i \neq x_j} \left\{ \frac{(x_j - x_i)^T (f_{ij}(x_i, x_j) - f_{ij}(x_i, x_i))}{\|x_j - x_i\|^2} \right\} \ge \epsilon > 0 \quad (27)$$

5) *the maximum flow is less than one, i.e.*

$$\sup_{i,\, x_i} \left\{ \sum_{j \in \mathcal{N}_i^{\rightarrow}} \frac{\|f_{ij}(x_i, x_j) - f_{ij}(x_i, x_i)\|}{\|x_j - x_i\|} \right\} < 1. \quad (28)$$

The consensus set $\Omega(\bar{x}(0)) = \{x | x_i = \bar{x}(0), \forall i \in V(G)\}$ is globally exponentially stable, where $x = [x_1^T, \ldots, x_n^T]^T$ is the stacked state vector and $\bar{x}(0) = \frac{1}{n}\sum_{i=1}^n x_i(0)$, $n = |V(G)|$.

Proof. In the following we use the notation introduced in the proof of Thm VI.2. Thus, the dynamics can be written as

$$x(t+1) = (P(x(t)) \otimes I_N) x(t) \quad (29)$$

$$\delta(t+1) = (P(\delta(t)) \otimes I_N) \delta(t), \quad (30)$$

where $P(x) = I_n - L$ is the Perron matrix, and $P(x) = P(\delta)$.

For any fixed $\delta \in \mathbb{R}^{n\times N}$ such that $(\mathbf{1}_n^T \otimes I_N)\delta = 0$, we have that $P(\delta)$ is a nonnegative doubly stochastic matrix. The strong positivity assumption (27) is equivalent to $k_{ij}(a) \ge \epsilon$ for all $a \ge 0$ and $(i, j) \in E(G)$, which trivially implies that all off-diagonal elements of $P(\delta)$ are non negative. Moreover, the maximum flow assumption can be restated as $\sum_{j\in N_i} k_{ij}(\|\delta_i - \delta_j\|) < 1$ which is equivalent to $P_{ii}(\delta) > 0$. The forward-backward consistency property implies that $P(\delta)$ is a normal matrix, for all δ. The Perron matrix $P(\delta)$ is double stochastic, i.e., G is balanced, because $\mathbf{1}_n$ is a right eigenvector of $L(\delta)$ and $P(\delta)P^T(\delta)\mathbf{1}_n = P^T(\delta)P(\delta)\mathbf{1}_n = P^T(\delta)\mathbf{1}_n$ which implies that $P^T(\delta)\mathbf{1}_n = a\mathbf{1}_n$, $a \neq 0$, is an eigenvector of $P(\delta)$ corresponding to the eigenvalue 1. Since $P^T(\delta)$ has the same spectrum as $P(\delta)$, it follows that a must be 1.

The Perron matrix is a contraction on the linear space defined by $(\mathbf{1}_n^T \otimes I_N)\delta = 0$, because

$$\|(P(\delta)\otimes I_N)\alpha\|^2 = \alpha^T(P(\delta)\otimes I_N)^T(P(\delta)\otimes I_N)\alpha \quad (31)$$
$$= \alpha^T((P(\delta)^T P(\delta))\otimes I_N)\alpha = \alpha^T((UD^*DU^*)\otimes I_N)\alpha \quad (32)$$
$$\leq |\mu_2(P(\delta))|^2 \cdot \|\alpha\|^2 , \quad (33)$$

where $\mu_2(P(\delta))$ is the second largest eigenvalue in absolute value of $P(\delta)$, $P(\delta) = UDU^*$, U is a unitary matrix, D is the diagonal matrix corresponding to the spectrum of $P(\delta)$, and * is the conjugate transpose operator. The inequality in (33) follows from the Courant-Fisher Theorem [26].

Lastly, it follows that $P(\delta(t))\otimes I_N$ is a sequence of contraction maps. All of them admit 0 as a fixed point. The Lipschitz constant for all of them is $q = \sup_{t\geq 0}|\mu_2(P(\delta(t)))| < 1$, because the strong positivity assumption guarantees that the entries of $P(\delta(t))$ can not become arbitrarily small as t goes to infinity. By Lemma VII.1 it follows that $\delta(t)$ converges to 0, where $X \subset \mathbb{R}^N$ is the space defined by $(I_N \otimes D_\alpha)\delta = 0$ with distance function induced by the Euclidean norm $\|\cdot\|$. □

B. Higher-order dynamics

In this section we extend the notion of $SE(N)$-invariance to higher-order pairwise interaction systems, i.e. each agent's dynamics has order $m \geq 2$. If the dynamics of these systems depends only the agents' states, then the definitions and results from Sec. III all hold. However, we are interested in systems whose dynamics depend on the agents' (generalized) velocities as well. For this class of systems, we show a similar result to Thm. III.7. As in Sec. V, all (generalized) velocities are measured with respect to a global inertial frame, but are represented in a reference frame of the agents' choice.

Definition VII.6 ($SE(N)$-invariant function)**.** *A function $f : \mathbb{R}^{Np\times m} \to \mathbb{R}^N$ is said to be $SE(N)$-invariant if for all $R \in SO(N)$ and all $w \in \mathbb{R}^N$ the following condition holds:*

$$Rf(x, v^1, \ldots, v^{m-1}) = f(\mathbf{R}x+\mathbf{w}, \mathbf{R}v^1, \ldots, \mathbf{R}v^{m-1}), \quad (34)$$

where $x, v^1, \ldots, v^{m-1} \in \mathbb{R}^{Np}$, $\mathbf{R} = R\otimes I_p$ and $\mathbf{w} = w\otimes \mathbf{1}_p$.

Definition VII.7 (Pairwise Interaction System)**.** *A continuous-time pairwise interaction system is a pair (G, F), where G is a graph and $F = \{(f_{ij}^0, \ldots, f_{ij}^{m-1}) \mid f_{ij}^r : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N, (i,j) \in E(G)\}$ is a set of tuple of functions associated to its edges. Each $i \in V(G)$ labels an agent, and a directed edge (i,j) indicates that node i interacts with (measures the state and velocities of) node j. The dynamics of each agent are described by*

$$x_i^{(m)} = \sum_{r=0}^{m-1} \sum_{j\in\mathcal{N}_i^{\rightarrow}} f_{ij}^r(x_i^{(r)}, x_j^{(r)}), \quad (35)$$

where $x_i^{(0)} = x_i$ and f_{ij}^r, $0 \leq r < m$, define the influence (interaction) of j on i.

We denote the *total interaction* on agent $i \in V(G)$ by

$$S_i(v^0 = x, v^1, \ldots, v^{m-1}) = \sum_{r=0}^{m-1} \sum_{j\in\mathcal{N}_i^{\rightarrow}} f_{ij}^r(v_i^r, v_j^r).$$

The definitions of $SE(N)$-invariant systems and quasi-linear systems remain unchanged, but are interpreted using the extended notions. The main theorem can be extended as follows:

Theorem VII.8. *Let (G, F) be a continuous-time pairwise interaction system such that $f_{ij}^r(v_i^r, v_j^r) = g_{ij}^r(v_i^r - v_j^r)$, where $g_{ij}^r : \mathbb{R}^N \to \mathbb{R}^N$ and $r \in \{1, \ldots, m-1\}$. Then (G, F) is $SE(N)$-invariant if and only if it is quasi-linear.*

Proof. Let S_i be the total interaction function of agent $i \in V(G)$. Let $v^r = 0$ for all $1 \leq r \leq m-1$. Since $RS_i(x, 0, \ldots, 0) = S_i(\mathbf{R}x + \mathbf{w}, 0, \ldots, 0)$ for all $(R, w) \in SE(N)$, we have by Lemma V.3 that $S_i(x, 0, \ldots, 0) = \sum_{j\in\mathcal{N}_i^{\rightarrow}} k_{ij}^0(\|x_i - x_j\|)(x_j - x_i)$. Similarly, let $x = 0$ and $v^r = 0$ for $r \neq s$, $1 \leq r, s \leq m-1$. For all $(R, w) \in SE(N)$, we have $RS_i(0, 0, \ldots, v^s, \ldots, 0) = R\sum_{j\in\mathcal{N}_i^{\rightarrow}} f_{ij}^s(v_i^s, v_j^s) = \sum_{j\in\mathcal{N}_i^{\rightarrow}} f_{ij}^s(Rv_i^s, Rv_j^s) = \sum_{j\in\mathcal{N}_i^{\rightarrow}} g_{ij}^s(Rv_i^s + w - (Rv_j^s + w)) = S_i(0, \ldots, \mathbf{R}v^s + \mathbf{w}, \ldots, 0)$. Again, by Lemma V.3 it follows that $S_i(0, 0, \ldots, v^s, \ldots, 0) = \sum_{j\in\mathcal{N}_i^{\rightarrow}} k_{ij}^s(\|v_j^s - v_i^s\|)(v_j^s - v_i^s)$. Overall, it follows that $S_i(v^0 = x, v^1, \ldots, v^{m-1}) = \sum_{r=0}^{m-1}\sum_{j\in\mathcal{N}_i^{\rightarrow}} k_{ij}^r(\|v_i^r - v_j^r\|)(v_j^r - v_i^r)$. Conversely, if all total interaction functions are quasi-linear, then it follows that the system is $SE(N)$-invariant. □

C. Switching topologies

The paper's main result, Thm. III.7, and the extensions to discrete-time and higher order systems hold also when the interaction topology G switches due to time-dependent signals. Intuitively, the time-varying topology is not related to the agents' reference frames. Thus, $SE(N)$-invariance implies the quasi-linear structure regardless of the topology of the system.

VIII. Examples

This section provides examples to clarify and illustrate the notions of $SE(N)$-invariance and quasi-linearity for pairwise interaction systems. We also consider existing pairwise multi-agent systems that have been studied in the literature, Table I. Many of these are $SE(N)$-invariant, one an example is not, and one is $SE(N)$-invariant only under certain conditions.

The following example shows an $SE(N)$-invariant system with local interaction functions which are not quasi-linear. However, as shown by Thm. V.4, the total interaction functions associated with the system's agents can be rewritten as sums of quasi-linear functions. Moreover, Ex. VIII.1 provides an example of a weakly stable system where the agents follow elliptical periodic orbits (see Fig. 3). The shape of the elliptical orbits depends on the agents' initial states: (1) equidistant initial states generate circular periodic trajectories (see Fig. 3(a)); (2) otherwise periodic elliptical trajectories are obtained (see Fig. 3(b)). This example, together with the systems considered in [12] and [13], show that $SE(N)$-invariant pairwise interaction systems have rich asymptotic behaviors aside from consensus.

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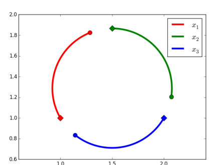
Example VIII.1. *Let (G, F) be a pairwise interaction system where $G = K_3$ is the complete graph with 3 vertices and*

$$f_{ij}(x_i, x_j) = \begin{cases} x_j & (i,j) \in \{(1,2),(2,3),(3,1)\} \\ -x_j & \text{otherwise} \end{cases}$$

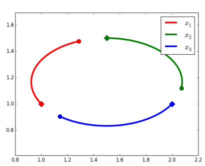
The pairwise interaction functions of this system are not quasi-linear in $x_j - x_i$, $(i,j) \in E(G)$. However, the system can easily be checked to be $SE(N)$-invariant. For agent 1 we have

$$\begin{aligned} S_1(x_1,x_2,x_3) &= f_{12}(x_1,x_2) + f_{13}(x_1,x_3) = x_2 - x_3 \\ RS_1 &= Rx_2 + w - (Rx_3 + w) \\ &= f_{12}(Rx_1 + w, Rx_2 + w) + f_{13}(Rx_1 + w, Rx_3 + w) \\ &= S_1(Rx_1 + w, Rx_2 + w, Rx_3 + w), \end{aligned}$$

where $R \in SO(N)$ and $w \in \mathbb{R}^N$. However, by Thm. V.4 the total interaction function S_1 must be a sum of quasi-linear functions. Indeed, we can rewrite $S_1 = x_2 - x_1 + (-1)(x_3 - x_1)$. Similarly, the $SE(N)$-property holds for the total interaction functions of the other two agents and these functions can be rewritten as sums of quasi-linear functions.



(a) $x_1^T(0) = [1,1]$, $x_2^T(0) = [\frac{3}{2}, 1 + \frac{\sqrt{3}}{2}]$, $x_3^T(0) = [2,1]$



(b) $x_1^T(0) = [1,1]$, $x_2^T(0) = [\frac{3}{2}, \frac{3}{2}]$, $x_3^T(0) = [2,1]$

Fig. 3. Trajectories of the $SE(2)$-invariant system presented in Ex. VIII.1. The three agents are shown in red, blue and green, respectively. Their states at time $t = 0$ sec and $t = 1$ sec are marked by diamonds and dots, respectively.

Example 1 in Tab. I was proposed in [12] to model swarm aggregation and is a quasi-linear system because $g(\cdot)$ is a quasi-linear function. The system exhibits an asymptotic behavior where the agents aggregate (in finite time) within a hyper-ball and stay inside it forever [12]. The second [15], third [10] and fourth [11] examples define the agents' dynamics based on potential functions. Example 2 from [15] drives the agents towards some goal states which are encoded in the $\gamma_i()$ functions, while ensuring that the agents avoid each other and fixed and known obstacles using the $\beta_i()$ functions. The system is not quasi-linear, because the potential function whose gradient is used for navigation depends explicitly on the agents' states, as opposed to distances between agents' states, and thus its gradient cannot be a quasi-linear function. Therefore, the multi-agent system in example 2 is not $SE(N)$-invariant. On the other hand, example 4 [11] is quasi-linear, because the gradients of $\nabla_{x_i} V_{ij}(\cdot)$ are quasi-linear functions. We conclude that the system is $SE(N)$-invariant in the sense of Def. VII.7 by Th. VII.8 for higher order systems with generalized velocities. The system in example 3 is quasi-linear if and only if the dynamics of the virtual leaders $\widetilde{f}_p$ are sums of quasi-linear functions, $1 \le p \le m$. Example 5 corresponds to systems implementing consensus and formation control [8], [9], [13]. It is easy to see that these systems are quasi-linear and therefore $SE(N)$-invariant. The last example shows a system of n point masses which interact due to gravity. This system is also quasi-linear and thus exhibits $SE(N)$-invariance, a well known fact in Hamiltonian mechanics [7].

TABLE I
THE TABLE CONTAINS EXAMPLES OF NETWORKED SYSTEMS THAT ARE QUASI-LINEAR, EXCEPT FOR THE SECOND EXAMPLE AND POSSIBLY THE FOURTH. IT FOLLOWS THAT THE QUASI-LINEAR SYSTEMS BELOW ARE ALSO $SE(N)$-INVARIANT BY THM. III.7. ALL SYSTEMS HAVE n AGENTS AND THE STATE OF AGENT $i \in \{1, \ldots, n\}$ IS DENOTE BY x_i. IN THE THIRD EXAMPLE, $\widetilde{x}_p$ REPRESENTS THE STATE OF A VIRTUAL LEADER p. THE MAPS $V_I, V_h, V_{ij} : \mathbb{R}_{\ge 0} \to \mathbb{R}$ REPRESENT POTENTIAL FUNCTIONS. THE GRADIENT OF V WITH RESPECT TO x_i IS DENOTED BY $\nabla_{x_i} V$.

	System dynamics	Ref.	*QL*?
1	$\dot{x}_i = \sum_{j=1}^{n} g(x_i - x_j)$ $g(y) = -y\left(a - b\exp\left(-\lVert y\rVert^2 c^{-1}\right)\right)$	[12]	Yes
2	$\dot{x}_i = -\alpha \nabla_{x_i}\left(\frac{\gamma_i(x)}{(\gamma_i(x)^k + \beta_i(x))^{1/k}}\right)$	[15]	No
3	$\ddot{x}_i = -\sum_{j \neq i}^{n} \nabla_{x_i} V_I(\lVert x_i - x_j\rVert) - \sum_{k=0}^{m-1} \nabla_{x_i} V_h(\lVert x_i - \widetilde{x}_k\rVert)$ $\ddot{\widetilde{x}}_p = \widetilde{f}_p(x_j, \widetilde{x}_k, \dot{x}_j, \dot{\widetilde{x}}_k), 1 \le p \le m$	[10]	Yes or No.
4	$\ddot{x}_i = -\sum_{j \in \mathcal{N}_i^{\rightarrow}} \left(\nabla_{x_i} V_{ij}(\lVert x_i - x_j\rVert) + (\dot{x}_i - \dot{x}_j)\right)$	[11]	Yes.
5	$\dot{x}_i = u_i$ or $x_i(k+1) = x_i(k) + u_i$ $u_i = \sum_{j \in \mathcal{N}_i^{\rightarrow}} a_{ij}(x_i - x_j)$ or $u_i = \sum_{j \in \mathcal{N}_i^{\rightarrow}} (\lVert x_i - x_j\rVert^2 - d_{ij})(x_i - x_j)$	[8], [9], [13]	Yes
6	$\ddot{x}_i = \frac{1}{m_i} \sum_{j=1, j \neq i}^{n} \frac{G m_i m_j}{\lVert x_i - x_j\rVert^3}(x_j - x_i)$	[7]	Yes

IX. CONCLUSIONS

In this paper, we studied the $SE(N)$-invariance property of multi-agent, locally interacting systems. This property, which guarantees the independence of a system of global reference frames, implies that control laws can be computed and executed locally (i.e., in each agent's frame) using only local information available to the agent. This property is critical in applications in which information about a global reference frame cannot be obtained, e.g., in GPS-denied environments.

The main contribution of the paper is to fully characterize pairwise interaction systems that are $SE(N)$-invariant. We showed that pairwise interaction systems are $SE(N)$-invariant if and only if they have a special *quasi-linear* form. Because of the simplicity of this form, this result can impact ongoing research on design of local interaction laws. The result can also be used as quick test of $SE(N)$-invariance for networked systems. We also described a subset of $SE(N)$-invariant pairwise interaction systems that reach consensus by exploiting their quasi-linear structure. Finally, we extended the results to discrete-time and high-order systems and systems with time-dependent switching topologies. As in the continuous case, we proved the convergence to consensus for a subclass of discrete-time $SE(N)$-invariant pairwise interaction systems.

X. Appendix. The case $N = 2$

The difference between the cases $N = 2$ and $N \geq 3$ is due to commutativity of rotations. $SO(2)$ is Abelian, while $SO(N)$ for $N \geq 3$ is not.

All results in the paper carry over to the case $N = 2$, because $SO(2)$ and its centralizer are Abelian. In all theorems quasi-linear functions are replaced with similar functions from the centralizer of $SO(2)$. In the following, we provide the characterization of $C_T(SO(2))$, which supports our claim.

Proposition X.1. *The centralizer of $SO(2)$ with respect to T is the submonoid $\{(k_1(\|x\|)I_2 + k_2(\|x\|)J_2)\, x\}$, where $k_1, k_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $J_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.*

Proof. Let $x \in \mathbb{R}^2$, $x \neq 0$, and $u = \frac{x}{\|x\|}$. It follows that $R_u = u_1 I_2 + u_2 J_2 \in SO(2)$, $x = R_u \|x\| e_1$, and

$$\begin{aligned} f(x) &= R_u f(\|x\| e_1) = \frac{1}{\|x\|} \begin{bmatrix} x_1 f_1(\|x\| e_1) - x_2 f_2(\|x\| e_1) \\ x_2 f_1(\|x\| e_1) + x_1 f_2(\|x\| e_1) \end{bmatrix} \\ &\triangleq \begin{bmatrix} k_1(\|x\|) & -k_2(\|x\|) \\ k_2(\|x\|) & k_1(\|x\|) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \end{aligned}$$

where $k_1(\|x\|) \triangleq \frac{f_1(\|x\| e_1)}{\|x\|}$ and $k_2(\|x\|) \triangleq \frac{f_2(\|x\| e_1)}{\|x\|}$. The case $x = 0$ follows from Lemma IV.2. □

References

[1] F. Bullo and A. Lewis, *Geometric Control of Mechanical Systems*. New York: Springer, 2005.

[2] F. Bullo, N. Leonard, and A. Lewis, "Controllability and motion algorithms for underactuated lagrangian systems on lie groups," *IEEE Transactions on Automatic Control*, vol. 45, no. 8, pp. 1437–1454, 2000.

[3] F. Bullo and R. Murray, "Tracking for fully actuated mechanical systems: a geometric framework," *Automatica*, vol. 35, no. 1, pp. 17 – 34, 1999.

[4] E. Fiorelli, N. Leonard, P. Bhatta, D. Paley, R. Bachmayer, and D. Fratantoni, "Multi-AUV control and adaptive sampling in Monterey Bay," *IEEE J of Oceanic Engineering*, vol. 31, no. 4, pp. 935–948, 2006.

[5] R. W. Beard, J. Lawton, and F. Y. Hadaegh, "A coordination architecture for spacecraft formation control," *IEEE Transactions on control systems technology*, vol. 9, no. 6, pp. 777–790, 2001.

[6] E. Montijano, D. Zhou, M. Schwager, and C. Sagues, "Distributed formation control without a global reference frame," in *American Control Conference (ACC), 2014*, June 2014, pp. 3862–3867.

[7] K. Meyer and G. Hall, *Introduction to Hamiltonian Dynamical Systems and the N-body Problem*. New York: Springer, 2009.

[8] R. Olfati-Saber, J. Fax, and R. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, Jan 2007.

[9] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, June 2003.

[10] N. Leonard and E. Fiorelli, "Virtual leaders, artificial potentials and coordinated control of groups," in *IEEE Conference on Decision and Control*, vol. 3, 2001, pp. 2968–2973.

[11] H. Tanner, A. Jadbabaie, and G. Pappas, "Stable flocking of mobile agents part I: dynamic topology," in *IEEE Conference on Decision and Control*, vol. 2, Dec 2003, pp. 2016–2021.

[12] V. Gazi and K. Passino, "Stability analysis of swarms," *IEEE Transactions on Automatic Control*, vol. 48, no. 4, pp. 692–697, April 2003.

[13] J. Cortés, "Global and robust formation-shape stabilization of relative sensing networks," *Automatica*, vol. 45, no. 12, pp. 2754–2762, 2009.

[14] M. Egerstedt and X. Hu, "Formation constrained multi-agent control," *IEEE Transactions on Robotics and Automation*, vol. 17, no. 6, pp. 947–951, 2001.

[15] M. De Gennaro and A. Jadbabaie, "Formation control for a cooperative multi-agent system using decentralized navigation functions," in *American Control Conference*, Minneapolis, MN, June 2006, pp. 1346–1351.

[16] P. Ogren, E. Fiorelli, and N. Leonard, "Formations with a mission: Stable coordination of vehicle group maneuvers," in *Symposium on Mathematical Theory of Networks and Systems*, July 2002, p. 15.

[17] R. Olfati-Saber and R. Murray, "Distributed cooperative control of multiple vehicle formations using structural potential functions," in *IFAC World Congress*, vol. 15, no. 1, Barcelona, Spain, 2002, pp. 242–248.

[18] A. Nettleman and B. Goodwine, "Symmetries and Reduction for Multi-Agent Control," in *IEEE International Conference on Robotics and Automation*, May 2015, pp. 5390–5396.

[19] E. Justh and P. Krishnaprasad, "Equilibria and steering laws for planar formations," *Systems & Control Letters*, vol. 52, no. 1, pp. 25–38, 2004.

[20] F. Zhang, M. Goldgeier, and P. S. Krishnaprasad, "Control of Small Formations using Shape Coordinates," in *IEEE International Conference of Robotics and Automation*, Taipei, Taiwan, 2003, pp. 2510–2515.

[21] K. S. Galloway, E. W. Justh, and P. S. Krishnaprasad, "Symmetry and reduction in collectives: cyclic pursuit strategies," *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 469, no. 2158, 2013.

[22] U. Halder and B. Dey, "Biomimetic algorithms for coordinated motion: Theory and implementation," in *IEEE International Conference on Robotics and Automation (ICRA)*, May 2015, pp. 5426–5432.

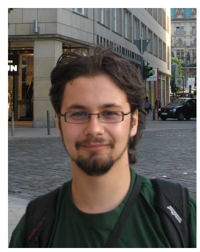
[23] D. G. Kendall, "Shape Manifolds, Procrustean Metrics, and Complex Projective Spaces," *Bulletin of the London Mathematical Society*, vol. 16, no. 2, pp. 81–121, 1984.

[24] C.-I. Vasile, M. Schwager, and C. Belta, "$SE(N)$ Invariance in Networked Systems," in *European Control Conference*, Linz, Austria, 2015.

[25] M. Artin, *Algebra*. Pearson Education, 2014.

[26] R. Horn and C. Johnson, *Matrix Analysis*. Cambridge University Press, 1990.

[27] E. E. Tyrtyshnikov, *A brief introduction to numerical analysis*. Springer, 2012.



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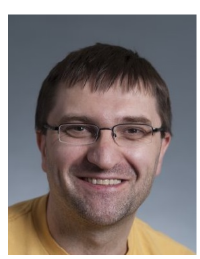

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