

# Consensus-based Distributed 3D Pose Estimation with Noisy Relative Measurements

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**Abstract**—In this paper we study consensus-based distributed estimation algorithms for estimating the global translation and rotation of each agent in a multi-agent system. We consider the case in which agents measure the noisy relative pose of their neighbors and communicate their estimates to agree upon the global poses in an arbitrary reference frame. The main contribution of this paper is a formal analysis that provides necessary and sufficient conditions to guarantee stability (in a Lyapunov sense) of the estimation system’s equilibria. We prove that consensus-based algorithms will diverge, even with arbitrarily small inconsistencies on the relative pose, unless the measurements satisfy *minimum consistency* conditions. We determine these consistency conditions for translation-only, rotation-only, and combined 3D pose estimation using the axis-angle rotation representation over undirected graphs. We then propose an initialization method based on these conditions that guarantees consistency and stability of the estimator’s equilibria. Additionally, we show that existing distributed estimation methods in literature exploit these conditions to guarantee convergence of their algorithms. Lastly, we perform simulations that show convergence when consistency conditions hold and divergence when they do not.

## I. INTRODUCTION

Consensus-based algorithms are often used for distributed estimation because they enable multiple decision-making agents to reach agreement on certain quantities by communicating their estimates over a graph [1]. These algorithms are preferable to centralized approaches when no centralized server exists or any given agent is too computationally constrained to compute the centralized solution for the entire network. This paper specifically considers nonlinear consensus-based algorithms for estimating the global pose of each agent in the multi-agent system in  $SE(3)$ , i.e., the Special Euclidean group in three dimensions (3D).

Prior work on distributed pose estimation generally considers relative pose measurements that are either (i) corrupted by noise or (ii) not corrupted by noise. Without noise, classic consensus algorithms [1] may be used for translation estimation. A number of methods have been proposed for orientation synchronization without noise that consider the axis-angle representation [2], QR-factorization [3], or synchronization on the manifold of  $SO(3)$ , the Special Orthog-

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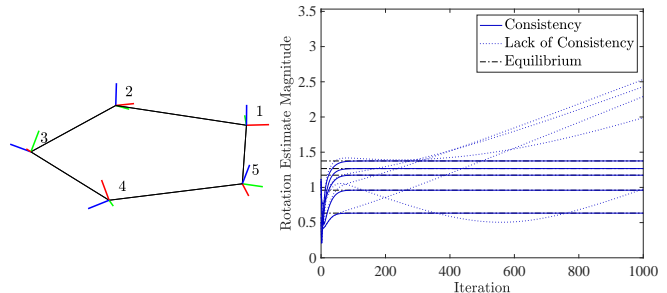


Fig. 1. We consider consensus-based estimation algorithms for estimating the global position and orientation of a multi-agent network such as the one shown above. On the left, agents (represented by coordinate frames) measure the noisy relative pose of their neighbors and communicate their estimates over a graph (black lines). We show the necessary and sufficient *consistency* conditions on these noisy measurements that guarantee stability of the distributed estimation method — illustrated on the right.

onal group in 3D [4]. A common-frame  $SE(3)$  estimation method is proposed in [5], where all agents desire to reach consensus on a common frame given a relative pose from that frame.

Noisy relative measurements have largely been considered in the analysis of translation-only estimation, a form of distributed linear estimation [6]. For example, [7] considers distance-only measurements while [8] considers bearing only measurements. The noisy relative translation measurements themselves are considered in [9] which discusses the distributed computation of the maximum likelihood translation estimate. The authors of [10] proposed solving for the centroid of the network and anchor-based translations simultaneously. Asynchronous estimation is treated in [11] and 2D estimation with Gaussian noise is analyzed in [12]. There are even fewer solutions that consider distributed 3D pose estimation given noisy relative measurements. Most notably, Tron et al. proposed a distributed gradient descent method that sequentially considers three separate objectives: first a convex relaxation of the rotation-only objective, then a translation-only objective, and finally the full 3D pose objective [13]. Recently, Thunberg et al. also studied distributed synchronization over all of  $SE(3)$  and guarantee convergence with an iterative algorithm that projects the rotation estimates at each step onto  $SO(3)$  [14]. These works consider distributed estimation for 3D pose but do not characterize the consistency conditions on the noisy measurements for consensus-based estimation systems as we do in this paper.

Inspired by distributed formation control [15], we consider

a general consensus-based distributed algorithm to compute the global pose estimate of all agents in the network. The agents first measure a corrupted version of the relative pose with respect their neighbors, and then run the algorithm using these measurements to agree upon their global poses in an arbitrary reference frame. Unfortunately, even an arbitrarily small disturbance to the relative pose over one link can lead to divergence of the estimate. Thus, the main contribution of the paper is a formal analysis of the necessary and sufficient conditions on the measurements to guarantee stability (in a Lyapunov sense) of the equilibria of a consensus-based estimation algorithm. We study the separate cases of translation-only (linear), rotation-only (nonlinear), and the coupled translation estimate from 3D pose estimation (non-linear). To guarantee convergence, the consensus protocol requires a minimum degree of *consistency* among the noisy relative pose measurements (see Fig. 1) in an undirected measurement graph. The characterization of these conditions yields a distributed initialization method for the estimation and also helps to explain update laws in existing distributed estimation algorithms [13], [14]. Simulation results corroborate the theoretical findings of the paper.

We present the notation and problem formulation in Section II followed by the consensus-based estimation algorithm in Section III. The conditions on the noisy measurements for convergence to the set of stable equilibria are proven in Section IV, followed by a discussion of the results in Section V. We present simulation results for a multi-agent system in Section VI and draw conclusions in Section VII.

## II. NOTATION AND PROBLEM SETUP

Consider  $n$  agents that are labeled by the set  $\mathcal{V} = \{1, \dots, n\}$ . The 3D rigid body pose of agent  $i \in \mathcal{V}$  is measured with respect to a global coordinate frame as a tuple containing a translation vector and a rotation matrix,  $(\mathbf{t}_i \in \mathbb{R}^3, \mathbf{R}_i \in SO(3))$ , where  $SO(3) = \{\mathbf{R} \in \mathbb{R}^{3 \times 3} \mid \mathbf{R}^T \mathbf{R} = \mathbf{I}_3, \det(\mathbf{R}) = +1\}$  and  $\mathbf{I}_3$  is a  $3 \times 3$  identity matrix.

We make use of the axis-angle representation of rotations defined as  $\mathbf{x}_i \in \mathbb{R}^3$ . The axis-angle may be thought of as a rotation  $\theta_i = \|\mathbf{x}_i\|_2$  about a unit axis  $\mathbf{u}_i = \frac{\mathbf{x}_i}{\|\mathbf{x}_i\|_2}$ . We further define the *hat*  $(\cdot)^\wedge : \mathbb{R}^3 \rightarrow so(3)$  and *vee*  $(\cdot)^\vee : so(3) \rightarrow \mathbb{R}^3$  operators, where  $so(3) = \{\mathbf{S} \in \mathbb{R}^{3 \times 3} \mid \mathbf{S}^T = -\mathbf{S}\}$  is the Lie algebra of the special orthogonal group  $SO(3)$ . The axis-angle is mapped to rotation matrices using the *exponential map*  $\exp(\mathbf{x}^\wedge) : so(3) \rightarrow SO(3)$ , or Rodrigues' rotation formula,

$$\exp(\mathbf{x}^\wedge) = \mathbf{I}_3 + \sin(\theta) \left( \frac{\mathbf{x}}{\theta} \right)^\wedge + (1 - \cos(\theta)) \left( \frac{\mathbf{x}}{\theta} \right)^\wedge^2,$$

where  $\mathbf{x} \in \mathbb{R}^3$ ,  $\mathbf{x}^\wedge \in so(3)$ , and  $\theta = \|\mathbf{x}\|_2$ . The inverse map is the *logarithmic map*  $\log(\mathbf{R}) : SO(3) \rightarrow so(3)$ ,

$$\log(\mathbf{R}) = \begin{cases} \frac{\phi}{2 \sin(\phi)} (\mathbf{R} - \mathbf{R}^T), & \text{if } \phi \neq 0 \\ 0, & \text{if } \phi = 0 \end{cases},$$

where  $\phi = \arccos\left(\frac{1}{2}(\text{Tr}(\mathbf{R}) - 1)\right)$ .

The agents can communicate and measure one another over a communication graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  with edges  $(i, j) \in \mathcal{E}$ . We assume this graph is undirected, implying bi-direction communication between connected agents, and that the topology is fixed with time. Agent  $i$  can communicate with all agents within its local neighborhood subgraph,  $\mathcal{N}_i \subset \mathcal{V}$ , which is a set of neighbor labels. These connections form the graph Laplacian [1],  $\mathbf{L} \in \mathbb{R}^{n \times n}$ . It is well known that for undirected graphs  $\mathbf{L}$  has eigenvalues  $\lambda_1 = 0 \leq \lambda_2 \leq \dots \leq \lambda_n$ . The eigenvector associated to  $\lambda_1$  is in the set of vectors with equal entries, i.e.,  $\lambda_1 \in \mathbf{1}_n = \{\mathbf{a} \in \mathbb{R}^n \mid \mathbf{a} = \alpha \mathbf{1}_n, \alpha \in \mathbb{R}\}$  [16], where  $\mathbf{1}_n$  represents the  $n$ -dimensional column vector of ones. This vector is then a left and right eigenvector of  $\mathbf{L}$  — as graph  $\mathcal{G}$  is undirected — and the null space of  $\mathbf{L}$  is  $Null(\mathbf{L}) = \mathbf{1}_n$ .

The relative pose over edge  $(i, j) \in \mathcal{E}$ , representing the relative pose of neighbor  $j$  measured with respect to agent  $i$ , is  $(\mathbf{t}_{ij}, \mathbf{R}_{ij}) = (\mathbf{R}_i^T(\mathbf{t}_j - \mathbf{t}_i), \mathbf{R}_i^T \mathbf{R}_j)$ . Each agent measures the noisy relative pose of their neighbors once at the beginning of the algorithm. This measurement is a perturbation of the true relative pose and is given by the tuple  $(\tilde{\mathbf{t}}_{ij}, \tilde{\mathbf{R}}_{ij}), \forall (i, j) \in \mathcal{E}$ . Although relative rotations are commonly measured in rotation matrix form, we consider the logarithmic map of such measurements and define the relative axis-angle measurement  $\tilde{\mathbf{x}}_{ij} = \log(\tilde{\mathbf{R}}_{ij})^\vee$ .

Each  $i^{\text{th}}$  agent maintains its own global pose estimate  $(\hat{\mathbf{t}}_i(t), \hat{\mathbf{x}}_i(t))$  at each time  $t \in \mathbb{R}$ . For clarity, we omit the time arguments in this paper and refer to the time-varying estimates as  $(\hat{\mathbf{t}}_i, \hat{\mathbf{x}}_i)$ . Further, denote  $\hat{\mathbf{R}}_i = \exp(\hat{\mathbf{x}}_i^\wedge)$ . It is important to note that although the estimates are time-varying, the measurements  $(\tilde{\mathbf{t}}_{ij}, \tilde{\mathbf{R}}_{ij})$  are not. This is because they are *measured* once at the beginning and used as constant offset terms in the estimation algorithm (see Section III). This is in contrast to typical formation control problems where agents measure their neighbor's relative pose at each time step and use this information as feedback in the control.

The goal of this paper is to find global translations and rotations that minimize error with respect to the noisy relative measurements as given by Problem 1.

**Problem 1** (3D Pose Estimation Problem).

$$\underset{\substack{\hat{\mathbf{t}}_1, \dots, \hat{\mathbf{t}}_n \\ \hat{\mathbf{x}}_1, \dots, \hat{\mathbf{x}}_n}}{\text{minimize}} \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} \|\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i - \tilde{\mathbf{x}}_{ij}\|_2^2 + \|\hat{\mathbf{t}}_j - \hat{\mathbf{t}}_i - \hat{\mathbf{R}}_i \tilde{\mathbf{t}}_{ij}\|_2^2.$$

Minimizing this objective in a distributed manner is challenging as it is non-convex due to the coupled nature of the positions and orientations. We will later discuss in Section V that the equilibria of our algorithm under the *consistency* conditions are local minima of Problem 1.

## III. CONSENSUS-BASED 3D POSE ESTIMATION

The consensus-based estimation problem we study is conceptually similar to distributed formation control, where the global position and orientation estimates of agent  $i$  evolve according to the following integrator dynamics,

$$\dot{\hat{\mathbf{t}}}_i = \boldsymbol{\nu}_i, \quad (1a)$$

$$\dot{\hat{\mathbf{x}}}_i = \mathbf{L}_{\hat{\mathbf{x}}_i} \boldsymbol{\omega}_i. \quad (1b)$$

The interaction matrix  $\mathbf{L}_{\hat{\mathbf{x}}}$  (note the difference from the graph Laplacian  $\mathbf{L}$ ) for an axis-angle vector  $\hat{\mathbf{x}} = \hat{\theta}\hat{\mathbf{u}} \in \mathbb{R}^3$  is given by Malis et al. [17] as,

$$\mathbf{L}_{\hat{\mathbf{x}}} = \mathbf{I}_3 + \frac{\hat{\theta}}{2}\hat{\mathbf{u}}^\wedge + \left(1 - \frac{1}{4} \frac{\sin(\hat{\theta})}{\sin\left(\frac{1}{2}\hat{\theta}\right)^2}\right)\hat{\mathbf{u}}^\wedge{}^2. \quad (2)$$

This matrix is defined as  $\mathbf{I}_3$  when  $\hat{\theta} = 0$  and is unique for an axis-angle if  $\hat{\mathbf{x}} \in \mathbb{S}_{2\pi} = \{\hat{\mathbf{x}} \in \mathbb{R}^3 \mid \|\hat{\mathbf{x}}\|_2 \in (0, 2\pi)\}$  [2]. In the estimator dynamics (1), the ‘‘control inputs’’ are  $\mathbf{v}_i \in \mathbb{R}^3$  and  $\mathbf{w}_i \in \mathbb{R}^3$  for the translation and rotation terms, respectively. These inputs are defined in our method as,

$$\mathbf{v}_i = \sum_{j \in \mathcal{N}_i} \hat{\mathbf{t}}_j - \hat{\mathbf{t}}_i - \exp(\hat{\mathbf{x}}_i^\wedge) \tilde{\mathbf{t}}_{ij}, \quad (3a)$$

$$\mathbf{w}_i = \sum_{j \in \mathcal{N}_i} \hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i - \tilde{\mathbf{x}}_{ij}. \quad (3b)$$

In the rest of this paper, we refer to system (1) with control inputs (3) as the *consensus-based SE(3) estimation system*. Note that this system is nonlinear due to the coupled quantities in the translation control and the interaction matrix that is a function of the rotation axis-angle vector.

**Remark 1** (2D pose variant). *The algorithm and analysis presented in this paper are for 3D pose, however the results are also applicable for 2D pose. In fact, the axis-angle dynamics in 2D update a scalar quantity (a yaw angle) since all rotation vectors are aligned.*

This estimation scheme takes a similar form to that of [15], however there are several major differences. First, we use the axis-angle representation for rotation consensus so that we may reason about the rotational estimate’s equilibria. Second, as we already mentioned, the measurements are observed once at the beginning of the estimation and not at each step in the estimation. This way, the noise is not fed back into the system at each time step. The other major difference is that the measurements are noisy and no longer encode the *desired formation* as in formation control literature. Consensus-based estimators with perturbed measurements like this are not guaranteed to converge to a stable equilibria set [6]. Hence, we rigorously prove in the following section the specific required structure of these offsets such that the estimator converges to a stable equilibria.

#### IV. ANALYSIS

We now state the necessary and sufficient conditions on the noisy measurements ( $\tilde{\mathbf{t}}_{ij}, \tilde{\mathbf{x}}_{ij}$ ) for stability of the proposed 3D pose estimation system’s equilibria set. We first review the translation-only case with constant, known rotations, and show the expression for the equilibria as a function of the noisy measurements, which in turn yields a condition on the noise for global exponential stability. We extend this result for the rotation only case with the axis-angle representation yielding a set of locally asymptotically stable equilibria. Finally, we inspect the coupled translation case (now including rotation estimates) and show stability for a

time-varying disturbance term that decays asymptotically. We note that much of the analysis in this section is performed on systems that are comprised of all agent’s estimations but the distributed estimation algorithm does not need this global information.

Before we begin, we introduce the notion of *consistency* and present three definitions. We later discuss the equilibria of our distributed estimation in terms of these definitions.

**Definition 1** (Global consistency). *The measurements are globally consistent if  $\tilde{\mathbf{R}}_{ij} = \tilde{\mathbf{R}}_{i\ell}\tilde{\mathbf{R}}_{\ell j}$  and  $\tilde{\mathbf{t}}_{ij} = \tilde{\mathbf{t}}_{i\ell} + \tilde{\mathbf{R}}_{i\ell}\tilde{\mathbf{t}}_{\ell j}$ , for all  $i, \ell, j \in \mathcal{V}$ . In other words, the relative pose from agent  $i$  back to agent  $i$  must be zero (identity) for any cycle in the graph.*

**Definition 2** (Minimal consistency). *The measurements are minimally consistent if  $\sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} \tilde{\mathbf{t}}_{ij} = \mathbf{0}_3$  and  $\sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} \tilde{\mathbf{x}}_{ij} = \mathbf{0}_n$ , where  $\mathbf{0}_n$  represents the  $n$ -dimensional column vector of zeros.*

**Definition 3** (Pairwise consistency). *The measurements are pairwise consistent if  $\tilde{\mathbf{R}}_{ij} = \tilde{\mathbf{R}}_{ji}^T$  and  $\tilde{\mathbf{t}}_{ij} = \tilde{\mathbf{R}}_{ij}\tilde{\mathbf{t}}_{ji}$ ,  $\forall (i, j) \in \mathcal{E}$ . This is a particular case of minimal consistency (Definition 2) that is easy to satisfy with real noisy measurements as we discuss in Section V.*

#### A. Equilibria of the Translation-only Estimate

To begin our analysis of the proposed estimation system, we first consider the translation estimation system with fixed rotation estimates. Specifically, we consider the translation-only estimate as system (1a) with control (3a),

$$\dot{\hat{\mathbf{t}}}_i = \sum_{j \in \mathcal{N}_i} \hat{\mathbf{t}}_j - \hat{\mathbf{t}}_i - \exp(\hat{\mathbf{x}}_i^\wedge) \tilde{\mathbf{t}}_{ij}, \quad (4)$$

where each  $\mathbf{x}_i$  is assumed to be known and constant.

For clarity of presentation, we consider the 1D version of system (4) and then extend the results to 3D. We stack the dynamics of each 1D agent into one vector  $\mathbf{t}$  with the dynamics,

$$\dot{\mathbf{t}} = -\mathbf{L}\mathbf{t} - \boldsymbol{\delta}, \quad (5)$$

where  $\mathbf{t} = [\hat{t}_1, \dots, \hat{t}_n]^T \in \mathbb{R}^n$  is the stacked vector of all translation estimates — we remove the hat on vectors of stacked estimates for simplicity in notation. In this system,  $\mathbf{L} \in \mathbb{R}^{n \times n}$  is the graph Laplacian, and  $\boldsymbol{\delta} \in \mathbb{R}^n$  is the vector of desired offsets where each component is  $\delta_i = \sum_{j \in \mathcal{N}_i} \tilde{t}_{ij}$ . The rotations are temporarily omitted when considering only 1D.

We are interested in analyzing first, the structure of  $\boldsymbol{\delta}$  such that the equilibria of (5) are stable, and second, the closed-form expression of the set of stable equilibria. We present the following proposition for stability that follows as a generalization of the classic consensus-based formation control results presented in [1].

**Proposition 1** (Scalar translation-only stability). *The set of equilibria for system (5),  $\mathbb{E}_{\mathbf{t}} = \{\mathbf{t} \in \mathbb{R}^n \mid \dot{\mathbf{t}} = \mathbf{0}_n\}$ , is globally exponentially stable if and only if the vector of offsets  $\boldsymbol{\delta}$  is minimally consistent.*

*Proof.* The proof is split into two parts: first, that minimally consistent  $\delta \implies$  stability and second, stability  $\implies \delta$  is minimally consistent.

First, let us assume that the vector  $\delta$  is minimally consistent, i.e.,  $\sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} \tilde{t}_{ij} = \mathbf{1}_n^T \delta = 0$ . We show that the set of equilibria is stable in a Lyapunov sense. Define a candidate Lyapunov function  $V(\mathbf{z}) = \frac{1}{2} \mathbf{z}^T \mathbf{z}$  that is positive definite on all  $\mathbb{R}^n$ , where  $\mathbf{z} = \mathbf{t} - \mathbf{t}_{eq}$  with  $\mathbf{t}_{eq} \in \mathbb{E}_{\mathbf{t}}$ . The time derivative of this function is,

$$\begin{aligned} \dot{V}(\mathbf{z}) &= \mathbf{z}^T \dot{\mathbf{z}} = \mathbf{z}^T (-\mathbf{L}\mathbf{t} - \delta) \\ &= -\mathbf{z}^T \mathbf{L}\mathbf{z} - \mathbf{z}^T (\mathbf{L}\mathbf{t}_{eq} + \delta). \end{aligned} \quad (6)$$

For stability, we require that  $\dot{V}(\mathbf{z}) < 0$  for any  $\mathbf{z} \in \mathbb{R}^n$  and is exactly zero at the origin  $\mathbf{z} = \mathbf{0}_n$ . Let us consider the second term of  $\dot{V}(\mathbf{z})$  which is  $-\mathbf{z}^T (\mathbf{L}\mathbf{t}_{eq} + \delta)$ . This term is always precisely zero by definition of  $\mathbf{t}_{eq} \in \mathbb{E}_{\mathbf{t}}$ , i.e.,  $(-\mathbf{L}\mathbf{t}_{eq} - \delta) = \mathbf{0}_n$  for any delta, including a minimally consistent one. Now, we are only concerned with the first term of  $\dot{V}(\mathbf{z})$  which is  $-\mathbf{z}^T \mathbf{L}\mathbf{z}$ . This term is known from standard linear consensus literature to be negative semi-definite with an invariant null space [1]. Moreover, the rate of change is upper bounded by the smallest eigenvalue of the Laplacian, implying that the set of equilibria is globally exponentially stable.

Second, assume that the set of equilibria is globally exponentially stable. We now show that the equilibria  $\mathbb{E}_{\mathbf{t}}$ , which are a function of  $\delta$ , also require minimal consistency. This assumption implies that  $\dot{V}(\mathbf{z})$  is precisely zero at the origin, i.e., when  $\mathbf{t} \in \mathbb{E}_{\mathbf{t}}$ . Additionally, this implies that  $\dot{\mathbf{t}} = -\mathbf{L}\mathbf{t}_{eq} - \delta = \mathbf{0}_n$ . Now, we inspect this equilibrium condition using the Singular Value Decomposition (SVD) of the graph Laplacian  $\mathbf{L} = \mathbf{U}\Sigma\mathbf{U}^T$ :

$$\mathbf{L} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \bar{\Sigma} & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{B}^T & \mathbf{D} \end{bmatrix},$$

where matrix  $\mathbf{A}$  is the upper-left  $(n-1) \times (n-1)$  block of matrix  $\mathbf{U}$  and vectors  $\mathbf{B}$  and  $\mathbf{C}$  follow from that definition —  $\mathbf{D}$  is a scalar in this case. Likewise,  $\bar{\Sigma}$  is the upper-left  $(n-1) \times (n-1)$  block of matrix  $\Sigma$ . Also, define the notation  $\mathbf{t}_{1:n-1}$  as the first  $n-1$  entries of a vector  $\mathbf{t}$  and  $\mathbf{t}_n$  as the last component of that same vector. The affine system (5) may then be expressed as  $\mathbf{U}^T \dot{\mathbf{t}} = -\Sigma \mathbf{U}^T \mathbf{t} - \mathbf{U}^T \delta$ . Define the coordinate transformation  $\bar{\mathbf{t}} = \mathbf{U}^T \mathbf{t}$  such that the equilibrium condition is,

$$\begin{aligned} \begin{bmatrix} \dot{\bar{\mathbf{t}}}_{1:n-1} \\ \dot{\bar{\mathbf{t}}}_n \end{bmatrix} &= - \begin{bmatrix} \bar{\Sigma} & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^T & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{t}}_{1:n-1} \\ \bar{\mathbf{t}}_n \end{bmatrix} \\ &\quad - \begin{bmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{B}^T & \mathbf{D} \end{bmatrix} \begin{bmatrix} \delta_{1:n-1} \\ \delta_n \end{bmatrix}. \end{aligned} \quad (7)$$

The equilibria of this new system are given by any  $\bar{\mathbf{t}}_n \in \mathbb{R}$  as long as  $\mathbf{B}^T \delta_{1:n-1} + \mathbf{D} \delta_n = 0$ . This condition is equivalent to  $[\mathbf{B}^T, \mathbf{D}] \delta = \mathbf{1}_n^T \delta = 0$  since the last column of  $\mathbf{U}$  is  $\lambda_1$ , the eigenvector that corresponds to the smallest eigenvalue. Thus, we arrive at the requirement that  $\delta$  must be minimally consistent given that the equilibria are stable.  $\square$

Although Proposition 1 is for a 1D system, we extend this result to 3D and present a corollary for the stability of the

3D equilibria. In 3D with fixed, known rotations, system (5) has the vectorized form,

$$\dot{\mathbf{t}} = -(\mathbf{L} \otimes \mathbf{I}_3) \mathbf{t} - \delta, \quad (8)$$

where the vector of estimates is now  $\mathbf{t} = [\hat{\mathbf{t}}_1^T, \dots, \hat{\mathbf{t}}_n^T]^T \in \mathbb{R}^{3n}$ ,  $(\mathbf{L} \otimes \mathbf{I}_3) \in \mathbb{R}^{3n \times 3n}$ ,  $\otimes$  represents the Kronecker product, and  $\delta \in \mathbb{R}^{3n}$  is the vector of noisy translation offsets with an  $i^{\text{th}}$  component  $\sum_{j \in \mathcal{N}_i} \exp(\mathbf{x}_i^\wedge) \tilde{\mathbf{t}}_{ij} \in \mathbb{R}^3$ .

**Corollary 2** (3D translation-only equilibria). *The system (8) with the equilibria set  $\mathbb{E}_{\mathbf{t}} = \{\mathbf{t} \in \mathbb{R}^{3n} \mid \dot{\mathbf{t}} = \mathbf{0}_{3n}\}$  is globally exponentially stable if and only if the vector of offsets  $\delta$  is minimally consistent.*

*Proof.* The proof follows directly from Proposition 1, where the the minimum consistency condition is on the vectorized offsets, i.e.,  $\sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} \exp(\mathbf{x}_i^\wedge) \tilde{\mathbf{t}}_{ij} = \mathbf{0}_3$ , and the set of equilibria is found using the SVD on the matrix  $(\mathbf{L} \otimes \mathbf{I}_3)$  instead of just  $\mathbf{L}$ .  $\square$

From the proof of Proposition 1, we see that the equilibria of (7) are given for any  $\bar{\mathbf{t}}_n \in \mathbb{R}$ . We see that if this condition is not satisfied, then  $\dot{\bar{\mathbf{t}}}_n \neq 0$  and the formation is in motion. This means that  $\bar{\mathbf{t}}_n$  is a degree of freedom in the solution corresponding to arbitrarily translating the stable formation in  $\mathbb{R}$ . Using this observation and Corollary 2, we state the following theorem for the closed-form expression of the 3D consensus equilibria as a function of this degree of freedom and the minimally consistent  $\delta$ . In general this is a set of equilibria, however from standard linear consensus we know that the equilibrium point of the estimate is the mean of the initial conditions of the estimate.

**Theorem 3** (Translation-only set of equilibria). *The set of stable equilibria for system (8) is,*

$$\mathbb{E}_{\mathbf{t}} = \mathbf{U}\Psi\mathbf{U}^T\delta, \quad (9)$$

where  $\mathbf{U}$  is defined from the SVD of  $(\mathbf{L} \otimes \mathbf{I}_3)$ ,  $\Psi = -\text{Diag}\left(\left[\frac{1}{\lambda_n} \mathbf{1}_3^T, \dots, \frac{1}{\lambda_2} \mathbf{1}_3^T, \bar{\mathbf{t}}_b^T\right]\right)$ ,  $\text{Diag}(\cdot)$  is the block diagonal operator that maps vectors to matrices, and  $\bar{\mathbf{t}}_b \in \mathbb{R}^3$  represents the degree of freedom in the 3D equilibria.

*Proof.* This solution follows from solving (7) for the equilibria  $\bar{\mathbf{t}}$  in 3D. To do so we must redefine some of the variables from the proof of Proposition 1. Consider the SVD,

$$\begin{aligned} (\mathbf{L} \otimes \mathbf{I}_3) &= \mathbf{U}\Sigma\mathbf{U}^T = \\ &\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \bar{\Sigma} & \mathbf{0}_{(3n-3) \times 3} \\ \mathbf{0}_{(3n-3) \times 3}^T & \mathbf{0}_{3 \times 3} \end{bmatrix} \begin{bmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{B}^T & \mathbf{D}^T \end{bmatrix}, \end{aligned}$$

where matrix  $\mathbf{A}$  is the upper-left  $(3n-3) \times (3n-3)$  block of matrix  $\mathbf{U}$  and matrices  $\mathbf{B}$ ,  $\mathbf{C}$ , and  $\mathbf{D}$  follow from that definition,  $\bar{\Sigma}$  is the upper-left  $(3n-3) \times (3n-3)$  block of matrix  $\Sigma$ , and  $\mathbf{0}_{n \times n}$  is the  $n \times n$  matrix of zeros.

The affine system (8) may be expressed as  $\mathbf{U}^T \dot{\mathbf{t}} = -\Sigma \mathbf{U}^T \mathbf{t} - \mathbf{U}^T \delta$ . We define for clarity the subscripts  $\mathbf{t}_a$  and  $\mathbf{t}_b$  to denote the first  $(3n-3)$  and the last three entries, respectively, of a vector  $\mathbf{t}$ . Define the same transform  $\bar{\mathbf{t}} =$

$\mathbf{U}^T \mathbf{t}$  with the following dynamics,

$$\begin{bmatrix} \dot{\bar{\mathbf{t}}}_a \\ \dot{\bar{\mathbf{t}}}_b \end{bmatrix} = - \begin{bmatrix} \bar{\Sigma} & \mathbf{0}_{(3n-3) \times 3} \\ \mathbf{0}_{(3n-d) \times 3}^T & \mathbf{0}_{3 \times 3} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{t}}_a \\ \bar{\mathbf{t}}_b \end{bmatrix} - \begin{bmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{B}^T & \mathbf{D}^T \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_b \end{bmatrix}. \quad (10)$$

The equilibria of this new system are defined by setting  $\dot{\bar{\mathbf{t}}} = \mathbf{0}_{3n}$ . We again see that  $\dot{\bar{\mathbf{t}}}_b = \mathbf{0}_3$  for any  $\bar{\mathbf{t}}_b \in \mathbb{R}^3$  as long as  $\mathbf{B}^T \delta_a + \mathbf{D}^T \delta_b = \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} \exp(\mathbf{x}_i^\wedge) \bar{\mathbf{t}}_{ij} = \mathbf{0}_3$  due to the eigenvector of  $\mathbf{U}$  associated to the smallest eigenvalue, i.e., the consistency condition in 3D.

Now, the first  $(3n - 3)$  components of (10) are  $\dot{\bar{\mathbf{t}}}_a = -\bar{\Sigma}^{-1}(\mathbf{A}^T \delta_a + \mathbf{C}^T \delta_b)$ . Further, invert the change of coordinates such that  $\mathbf{t} = \mathbf{U} \bar{\mathbf{t}}$ . Then, the equilibria set is,

$$\begin{aligned} \mathbb{E}_{\mathbf{t}} &= \mathbf{U} \begin{bmatrix} -\bar{\Sigma}^{-1}(\mathbf{A}^T \delta_a + \mathbf{C}^T \delta_b) \\ \bar{\mathbf{t}}_b \end{bmatrix} \\ &= \mathbf{U} \begin{bmatrix} -\bar{\Sigma}^{-1}[\mathbf{A}^T, \mathbf{C}^T] \delta \\ \bar{\mathbf{t}}_b [\mathbf{B}^T, \mathbf{D}^T] \delta \end{bmatrix} \\ &= \mathbf{U} \begin{bmatrix} -\text{Diag} \left( \left[ \frac{1}{\lambda_n} \mathbf{1}_3^T, \dots, \frac{1}{\lambda_2} \mathbf{1}_3^T \right] \right) & \mathbf{0}_{(3n-3) \times 3} \\ \mathbf{0}_{3 \times (3n-3)} & \text{Diag}(\bar{\mathbf{t}}_b) \end{bmatrix} \mathbf{U}^T \delta, \end{aligned} \quad (11)$$

and the statement follows.  $\square$

### B. Equilibria of the Rotation-only Axis-Angle Estimate

We next inspect the stability of the equilibria for the rotation estimate (1b) with control input (3b). Note this estimate is independent of the translation estimate. We consider the vectorized version of the axis-angle estimation dynamics by defining the following terms:

$$\begin{aligned} \mathbf{x} &= [\hat{\mathbf{x}}_1^T, \dots, \hat{\mathbf{x}}_n^T]^T \in \mathbb{R}^{3n}, \\ \delta_{\mathbf{x}} &= \left[ \sum_{j \in \mathcal{N}_1} \tilde{\mathbf{x}}_{1j}^T, \dots, \sum_{j \in \mathcal{N}_n} \tilde{\mathbf{x}}_{nj}^T \right]^T \in \mathbb{R}^{3n}, \text{ and} \\ \Gamma &= \text{BlkDiag}(\mathbf{L}_{\hat{\mathbf{x}}_1}, \dots, \mathbf{L}_{\hat{\mathbf{x}}_n}) \in \mathbb{R}^{3n \times 3n}. \end{aligned}$$

Here the  $\text{BlkDiag}(\cdot)$  operator maps a list of matrices to a block-diagonal matrix. Now, we are interested in the evolution of the stacked vector  $\mathbf{x}$  and note that this system can be expressed as a function of the previously defined terms,

$$\dot{\mathbf{x}} = -\Gamma(\mathbf{L} \otimes \mathbf{I}_3)\mathbf{x} - \Gamma \delta_{\mathbf{x}}. \quad (12)$$

Notice that the structure of the control input (3b) is similar in form to the standard translation-only system, however the axis-angle dynamics include the nonlinear interaction matrix  $\mathbf{L}_{\hat{\mathbf{x}}_i}$ . Similarly to Section IV-A, we state the necessary and sufficient conditions of  $\delta_{\mathbf{x}}$  required for stability of this system's equilibria. First, we present the following two lemmas regarding the null space and positive definiteness of the term  $\Gamma(\mathbf{L} \otimes \mathbf{I}_3)$  and then present the stability theorem.

**Lemma 4.** *If  $\mathbf{x} \in \mathcal{S}_{2\pi} = \{\mathbf{x} \in \mathbb{R}^{3n} \mid \hat{\mathbf{x}}_i \in \mathcal{S}_{2\pi}, \forall i \in \mathcal{V}\}$  then matrix  $\Gamma(\mathbf{L} \otimes \mathbf{I}_3)$  is full rank and has a null space equal to the set  $\mathbb{1}_{(n,3)} = \{\mathbf{a} \in \mathbb{R}^{3n} \mid \mathbf{a} = \mathbf{1}_n \otimes \boldsymbol{\alpha}, \boldsymbol{\alpha} = [\alpha_1, \alpha_2, \alpha_3]^T \in \mathbb{R}^3\}$ .*

*Proof.* Each  $\mathbf{L}_{\hat{\mathbf{x}}_i}$  is full rank since  $\hat{\mathbf{x}}_i \in \mathcal{S}_{2\pi}, \forall i \in \mathcal{V}$  [2]. This implies that the block diagonal  $\Gamma$  is full rank and the null space of interest is  $\text{Null}(\mathbf{L} \otimes \mathbf{I}_3)$ . From Lemma 2.2 in [18], the null space of  $(\mathbf{L} \otimes \mathbf{I}_3)$  is set of stacked vectors with repeated, constant components:  $\mathbb{1}_{(n,3)}$ .  $\square$

**Lemma 5.** *If  $\mathbf{x} \in \mathcal{S}_{\pi} = \{\mathbf{x} \in \mathbb{R}^{3n} \mid \hat{\mathbf{x}}_i \in \mathcal{S}_{\pi}, \forall i \in \mathcal{V}\}$  then matrix  $\Gamma(\mathbf{L} \otimes \mathbf{I}_3)$  has non-negative eigenvalues.*

*Proof.* Consider first the interaction matrix (2) for an agent  $i$  assuming  $\hat{\mathbf{x}}_i \in \mathcal{S}_{\pi}, \forall i \in \mathcal{V}$ . The second two matrices of the sum in (2) are functions of  $\hat{\theta}_i$  and skew-symmetric forms of the unit vector  $\hat{\mathbf{u}}_i$ . The scalar multiples of these two terms are  $\frac{\hat{\theta}_i}{2}$  and  $(1 - \frac{1}{4} \frac{\sin(\hat{\theta}_i)}{\sin(\frac{1}{2}\hat{\theta}_i)}) \in (0, 1)$ . From properties of skew-symmetric matrices, the eigenvalues of  $\hat{\mathbf{u}}_i^\wedge$  include a zero and one purely imaginary pair while the eigenvalues of  $\hat{\mathbf{u}}_i^{\wedge 2}$  include a zero and two repeated  $-1$  values. Further, (2) has at least one eigenvalue equal to 1 since  $\hat{\mathbf{u}}_i^\wedge$  and  $\hat{\mathbf{u}}_i^{\wedge 2}$  are rank 2. Now, the sum of the eigenvalues of (2) is equal to the sum of all eigenvalues of the three matrices since these matrices are Hermitian [19], which is  $\geq 1$ . This implies that the eigenvalues of (2) are all non-negative and that  $\Gamma$  is positive semi-definite. Recall that the eigenvalues of  $(\mathbf{L} \otimes \mathbf{I}_3)$  are also non-negative. Finally, from properties of products of positive semi-definite matrices, the eigenvalues of the product  $\Gamma(\mathbf{L} \otimes \mathbf{I}_3)$  are also non-negative.  $\square$

**Theorem 6** (Rotation-only axis-angle stability). *The set of equilibria for system (12) defined by  $\mathbb{E}_{\mathbf{x}} = \{\mathbf{x} \in \mathbb{R}^{3n} \mid \dot{\mathbf{x}} = \mathbf{0}_{3n}\}$  is locally asymptotically stable in region  $\mathbf{x} \in \mathcal{S}_{\pi}$  if and only if  $\delta_{\mathbf{x}}$  is minimally consistent.*

*Proof.* Similarly to Proposition 1, the proof is split into two parts: minimally consistent  $\delta_{\mathbf{x}} \implies$  stability and vice versa.

First, we assume that  $\delta_{\mathbf{x}}$  is minimally consistent and show that the set of equilibria is stable in a Lyapunov sense. Define a positive definite candidate Lyapunov function  $V(\mathbf{z}) = \frac{1}{2} \mathbf{z}^T \mathbf{z}$  where  $\mathbf{z} = \mathbf{x} - \mathbf{x}_{eq}$  and  $\mathbf{x}_{eq} \in \mathbb{E}_{\mathbf{x}}$ . Then,

$$\begin{aligned} \dot{V}(\mathbf{z}) &= \mathbf{z}^T \dot{\mathbf{z}} \\ &= -\mathbf{z}^T \Gamma[(\mathbf{L} \otimes \mathbf{I}_3)\mathbf{x} + \delta_{\mathbf{x}}] \\ &= -\mathbf{z}^T \Gamma(\mathbf{L} \otimes \mathbf{I}_3)\mathbf{z} - \mathbf{z}^T \Gamma[(\mathbf{L} \otimes \mathbf{I}_3)\mathbf{x}_{eq} + \delta_{\mathbf{x}}] \end{aligned}$$

The second term of  $\dot{V}(\mathbf{z})$  is zero  $\forall \mathbf{z} \in \mathbb{R}^{3n}$  because of the definition of  $\mathbf{x}_{eq}$  and since  $\Gamma$  is full rank from Lemma 4 (i.e.,  $\mathbf{x} \in \mathcal{S}_{\pi} \subset \mathcal{S}_{2\pi}$ ). As in Proposition 1, we are then only concerned with the first term of  $\dot{V}(\mathbf{z})$ , which is  $-\mathbf{z}^T \Gamma(\mathbf{L} \otimes \mathbf{I}_3)\mathbf{z}$ . This term is negative semi-definite when  $\mathbf{z} \notin \text{Null}(\Gamma(\mathbf{L} \otimes \mathbf{I}_3))$  from Lemma 5 since  $\hat{\mathbf{x}}_i \in \mathcal{S}_{\pi}, \forall i \in \mathcal{V}$ . However,  $\dot{V}(\mathbf{z})$  is zero when  $\mathbf{z} \in \text{Null}(\mathbf{L})$ . Fortunately, the null space  $\mathbf{z} \in \mathbb{1}_{(n,3)}$  is invariant and therefore, by LaSalle's invariance principle, the set of equilibria is locally asymptotically stable in the region defined by  $\mathbf{x} \in \mathcal{S}_{\pi}$ .

Second, we assume that the set of equilibria is locally asymptotically stable and show that these equilibria  $\mathbb{E}_{\mathbf{x}}$  that are a function of  $\delta_{\mathbf{x}}$  are minimally consistent. Then  $\dot{V}(\mathbf{z})$  is precisely zero at the origin, i.e., when  $\mathbf{x} \in \mathbb{E}_{\mathbf{x}}$ , implying  $\dot{\mathbf{x}} = \mathbf{0}_{3n}$  and  $(\mathbf{L} \otimes \mathbf{I}_3)\mathbf{x} + \delta_{\mathbf{x}} = \mathbf{0}_{3n}$  since  $\Gamma$  is invertible for  $\mathbf{x} \in$

$\mathcal{S}_\pi \subset \mathcal{S}_{2\pi}$ . In this case, we are interested in the equilibria of a system that takes the exact form of (8) considering  $\delta_x$  instead of  $\delta$ . Therefore, from Corollary 2,  $\delta_x$  must be minimally consistent given that the equilibria are stable.  $\square$

The closed-form set of equilibria for system (12) may also be computed using Theorem 3 for  $\mathbf{x} \in \mathcal{S}_\pi$ .

### C. Equilibria of the Coupled Translation Estimate

Finally, we inspect the coupled translation estimation system with time-varying rotation estimates. This may be applied to the translation estimation when considering the full 3D pose estimation system. First, we consider a generalized version of this system and introduce a time-varying disturbance term,  $\delta(t) \in \mathbb{R}^{3n}$ , that represents the coupled rotation estimates and translation measurements from system (1a) with control (3a),

$$\dot{\mathbf{t}} = -(\mathbf{L} \otimes \mathbf{I}_3)\mathbf{t} - \delta(t). \quad (13)$$

We first present the following proposition regarding this system's equilibria with a generic decaying  $\delta(t)$  term that follows from Theorem 3.8 of [15] with the extension of minimally consistent offsets in the limit.

**Proposition 7** (Coupled stability with generic  $\delta(t)$ ). *If the time-varying disturbance  $\delta(t)$  satisfies,*

$$\lim_{t \rightarrow \infty} \delta(t) = \delta_\infty, \quad (14)$$

where  $\delta_\infty$  is a bounded minimally consistent vector, then,

$$\lim_{t \rightarrow \infty} \mathbf{t} = \mathbf{t}_{eq}, \quad (15)$$

where  $\mathbf{t}_{eq} \in \mathbb{E}_{\mathbf{t}} = \{\mathbf{t} \in \mathbb{R}^{3n} \mid \dot{\mathbf{t}} = \mathbf{0}_{3n}\}$  is an equilibrium point of (13).

*Proof.* Suppose the time-varying disturbance term  $\delta(t)$  satisfies (14) with  $\delta_\infty$  minimally consistent. Define the error term  $\mathbf{z} = \mathbf{t} - \mathbf{t}_{eq}$ . The time derivative is,

$$\begin{aligned} \dot{\mathbf{z}} &= -(\mathbf{L} \otimes \mathbf{I}_3)\mathbf{z} - \delta(t) \\ &= -(\mathbf{L} \otimes \mathbf{I}_3)\mathbf{z} + [-(\mathbf{L} \otimes \mathbf{I}_d)\mathbf{t}_{eq} - \delta(t)] \\ &= -(\mathbf{L} \otimes \mathbf{I}_3)\mathbf{z} + \beta. \end{aligned} \quad (16)$$

Consider the unforced system  $\dot{\mathbf{z}} = -(\mathbf{L} \otimes \mathbf{I}_3)\mathbf{z}$  and a positive definite candidate Lyapunov function  $V(\mathbf{z}) = \frac{1}{2}\mathbf{z}^T\mathbf{z}$ . Then,  $\dot{V}(\mathbf{z}) = -\mathbf{z}^T(\mathbf{L} \otimes \mathbf{I}_3)\mathbf{z} \leq -\lambda_2\|\mathbf{z}\|_2^2$ , where  $\lambda_2$  is the second smallest eigenvalue of  $\mathbf{L}$  [1]. This evaluates to zero when  $\mathbf{z} \in \mathbb{1}_{(n,3)}$  and as in Theorem 6, this set is invariant, implying that the equilibria set is globally asymptotically stable. Further, the inequality implies global exponential stability from Theorem 4.10 of [20]. Now, we turn our attention to the affine control input  $\beta$ . From Corollary 2 and since  $\delta_\infty$  is minimally consistent we know that  $\beta \rightarrow [-(\mathbf{L} \otimes \mathbf{I}_3)\mathbf{t}_{eq} - \delta_\infty] = \mathbf{0}_{nd}$  as  $t \rightarrow \infty$ . Thus, the control in the forced system  $\beta$  vanishes asymptotically and from Lemma 4.6 of [20], the system is input-to-state stable, yielding the stated result in (15).  $\square$

In our 3D pose estimation system, the vectorized coupled translation estimate takes the following form,

$$\dot{\mathbf{t}} = -(\mathbf{L} \otimes \mathbf{I}_3)\mathbf{t} - \delta_t, \quad (17)$$

where  $\delta_t \in \mathbb{R}^{3n}$  is the translation offset vector and each  $i^{\text{th}}$  three-dimensional sub-vector is given by  $\sum_{j \in \mathcal{N}_i} \exp(\hat{\mathbf{x}}_i^\wedge) \tilde{\mathbf{t}}_{ij}$ .

**Corollary 8** (Coupled translation stability). *If the distributed pose estimate system (1) is controlled by (3), the rotation measurements are minimally consistent, the coupled rotation-translation offsets obey  $\lim_{t \rightarrow \infty} \delta(t) = \delta_\infty$  with  $\delta_\infty$  minimally consistent, and the initial rotation estimates satisfy  $\mathbf{x} \in \mathcal{S}_\pi$ , then,*

$$\lim_{t \rightarrow \infty} \mathbf{t} = \mathbf{t}_{eq}, \quad (18)$$

where  $\mathbf{t}_{eq} \in \mathbb{E}_{\mathbf{t}} = \{\mathbf{t} \in \mathbb{R}^{3n} \mid \dot{\mathbf{t}} = \mathbf{0}_{nd}\}$  is an equilibrium point of (17).

*Proof.* The proof follows directly from Proposition 7 by considering the general form of system (13) with local stability given by the set of  $\mathbf{x} \in \mathcal{S}_\pi$ .  $\square$

## V. DISCUSSION

The result from the coupled translation estimation (Corollary 8) tells us that the translation and rotation estimation can be run concurrently. Previous distributed estimation works often consider these in separate, sequential algorithms, however this is not necessary with our method. Additionally, Corollary 8 requires that the coupled rotation-translation offsets tend towards a minimally consistent vector. How exactly that consistency is achieved is a design element of the algorithm and may be accomplished in many ways. We propose a number of options here and relate them to other algorithms in literature.

Since rotation is independent of translation, an intuitive technique is to average the initial pairwise rotation measurements between connected neighbors. The rotation measurements may be initialized at the beginning of the algorithm for each edge pair  $(i, j) \in \mathcal{E}$  by,

$$\tilde{\mathbf{R}}'_{ij} = \tilde{\mathbf{R}}_{ij} \exp\left(\frac{1}{2} \log(\tilde{\mathbf{R}}_{ij}^T \tilde{\mathbf{R}}_{ji}^T)\right), \quad (19)$$

such that  $\tilde{\mathbf{R}}'_{ij} \tilde{\mathbf{R}}'_{ji} = \mathbf{I}_3$ . This operation is not expensive as it involves one communication round between agent  $i$  and its immediate neighbors.

For translation, one must consider the time-varying rotation estimate. One approach is to average the coupled translation terms between each update loop using,

$$\tilde{\mathbf{t}}'_{ij} = \frac{1}{2}(\tilde{\mathbf{t}}_{ij} - \tilde{\mathbf{R}}'_{ij} \tilde{\mathbf{t}}_{ji}). \quad (20)$$

This operation only requires a similar one-hop communication at the beginning, however agent  $i$  can perform this operation without additional communication at all steps in the future. This is the technique used in Section VI.

In the algorithm by Tron et al. [13], the estimates are updated using the gradient of an objective similar to the one in Problem 1. Notably, these gradient-based updates naturally enforce pairwise consistency of the noisy measurements and thus satisfy our requirement for stability. Inspired by their

method, this real-time consistency can be incorporated into a modified control law for the estimation system (1):

$$\nu_i = \sum_{j \in \mathcal{N}_i} (\hat{\mathbf{t}}_j - \hat{\mathbf{t}}_i) + \frac{1}{2} \left( \hat{\mathbf{R}}_j \tilde{\mathbf{t}}_{ji} - \hat{\mathbf{R}}_i \tilde{\mathbf{t}}_{ij} \right). \quad (21)$$

This type of pairwise averaging also appears in [14] (Algorithm 1) despite the different form of rotation. We also relate our findings to formation control literature such as [15], where they require global consistency on the desired poses for the proof of convergence of the coupled translation and rotation formation. The global consistency requirement is significantly more restrictive than minimal consistency.

Our algorithm is not explicitly performing distributed gradient descent on the centralized objective (1), however we briefly make the connection to centralized  $SE(3)$  maximum likelihood estimation [21] that considers a similar objective.

**Proposition 9.** *The equilibria of the distributed estimation system given minimally consistent measurements are local minima for Problem (1).*

*Proof.* Denote the objective of Problem (1) as  $J$ . The gradients with respect to the estimates are,

$$\begin{aligned} \frac{\partial J}{\partial \hat{\mathbf{t}}_i} &= \sum_{j \in \mathcal{N}_i} 2(\hat{\mathbf{t}}_i - \hat{\mathbf{t}}_j) + \hat{\mathbf{R}}_i \tilde{\mathbf{t}}_{ij} - \hat{\mathbf{R}}_j \tilde{\mathbf{t}}_{ji}, \\ \frac{\partial J}{\partial \hat{\mathbf{x}}_i} &= \sum_{j \in \mathcal{N}_i} 2(\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_j) + \tilde{\mathbf{x}}_{ij} - \tilde{\mathbf{x}}_{ji} + \frac{\partial \hat{\mathbf{R}}_i}{\partial \hat{\mathbf{x}}_i} (\hat{\mathbf{t}}_j - \hat{\mathbf{t}}_i - \hat{\mathbf{R}}_i \tilde{\mathbf{t}}_{ij}). \end{aligned}$$

At equilibrium, under consistency in the measurements, both gradients evaluate to zero as each is a function of the estimation system multiplied by a constant. The extra offset terms in the gradients can also be considered as real-time averaging terms such as the ones in (21) which are pairwise consistent.  $\square$

## VI. NUMERICAL EXAMPLES

We demonstrate the results from Section IV with two sets of simulations: one that inspects the effects of consistency (Fig. 2) and one that illustrates convergence to the set of equilibria (Fig. 3). In each simulation we consider two cases:

- 1) translation-only estimation with known, fixed rotations (Section IV-A), and
- 2) full 3D pose estimation (Sections IV-B and IV-C).

In case 1, the fixed rotations are simply set to the ground truth rotations. Note that case 2 considers both rotation-only and coupled-translation estimation. In all examples, we consider  $n = 5$  agents for simplicity with global positions in a circular configuration as depicted in Fig. 1 with randomly sampled orientations in  $SO(3)$  [22]. The agents are connected in a ring communication graph which is depicted by black lines.

The initial estimates for each agent are initialized via randomly sampling from uniform distributions. The support is defined by  $\hat{\mathbf{t}}_i(0) \sim [-1, 1]$  for translation and  $\|\hat{\mathbf{x}}_i(0)\|_2 \sim [0, \pi]$  for rotation such that the conditions for Lemma 5 are satisfied. The rotation axis itself is a random unit vector in  $\mathbb{R}^3$ . The noisy translation measurements are corrupted by zero-mean Gaussian noise with covariance  $\mathbf{Q}_{\mathbf{t}_{ij}} = 0.1\mathbf{I}_3$

and the rotation measurements are corrupted by Langevin noise with covariance  $\mathbf{Q}_{\mathbf{R}_{ij}} = 10\mathbf{I}_3$  degrees<sup>2</sup>.

The first set of simulations in Fig. 2 demonstrate divergence of the estimation if the consistency conditions are not satisfied, even under small perturbations. We first enforce pairwise consistency for all edges except one (agent 1's measurement of agent 2) where we disturb this measurement by a 2.5% increase in magnitude (solid lines). We then consider the same scenario except we do not enforce any consistency in the network (dotted lines). Both scenarios lead to divergent estimations in both rotation and translation. It is interesting to note that the single inconsistent edge case diverges more quickly than the completely inconsistent case. The closed-form equilibria that should be reached according to Theorem 3 (under minimal consistency) are plotted for visualization (dashed lines).

The second set of simulations in Fig. 3 enforces minimal consistency and demonstrates the convergence of the estimate to the set of equilibria. The equilibrium point from Theorem 3 is also plotted (dashed lines). Before running the estimation for case 1 in Fig. 3(a), minimal consistency on the noisy translation measurements is achieved by averaging the measurements between each pair of agents that contains an edge, i.e., pairwise consistency from (20). For case 2 in Figs. 3(b) and 3(c), minimal consistency on the noisy rotation measurements is first achieved by averaging the measurements between pairs of agents containing an edge as in (19). Pairwise consistency is enforced on the noisy translation measurements upon each iteration of the estimation using (20). This step is similar to the update rule proposed in [13]. As expected, the 3D pose estimation converges to the equilibrium point under these minimally consistent measurement conditions.

## VII. CONCLUSION

We have shown the necessary and sufficient conditions for stability of the set of equilibria for a distributed 3D pose estimation algorithm based on the consensus protocol. This result proves that the noisy measurements must be minimally consistent in order for the equilibria of a consensus-based estimator to be stable. Specifically, we have shown stability and the consistency condition for translation-only, nonlinear rotational-only estimation, and the coupled translation estimation using distributed estimation. The results are applicable not only to distributed estimation as in our setting but to any affine consensus law, such as formation control. Future work includes proving similar results for rotation using other representations such as the rotation matrix or quaternion as well as different rotation objectives such as the geodesic distance.

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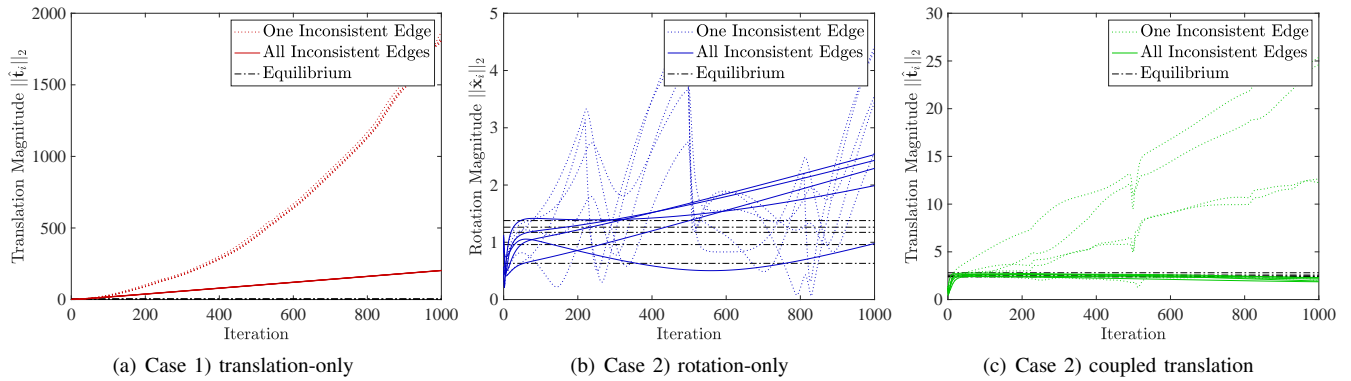


Fig. 2. Divergence results without measurement consistency for the five-agent distributed network considering inconsistent measurements for case 1 of translation-only estimation (a) and case 2 of 3D pose estimation — rotation-only (b) and coupled translation (c). The L2-norm of each  $i^{\text{th}}$  estimation vector is plotted to illustrate the lack of convergence. One inconsistent edge indicates that all edges are pairwise consistent except for the edge from agent 1 to agent 2. All inconsistent edges indicate that no edges are consistent. Interestingly, the single inconsistent edge case leads to faster divergence.

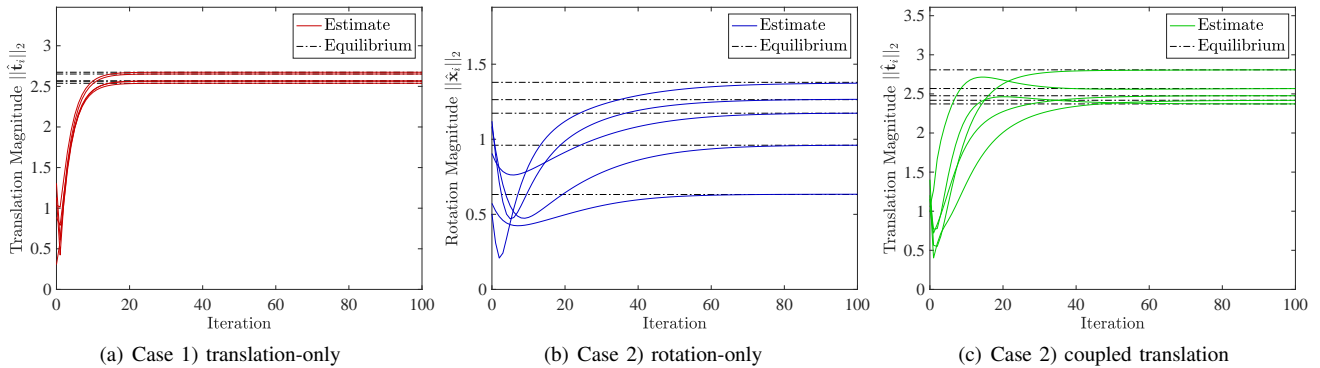


Fig. 3. Convergence results under measurement consistency for the five-agent distributed network considering case 1 of translation-only estimation (a) and case 2 of 3D pose estimation — rotation-only (b) and coupled translation (c). The L2-norm of each  $i^{\text{th}}$  estimation vector is plotted. The estimates converge to the known equilibrium point (Theorem 3) when the measurements are minimally consistent.

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